Divergence Groups Have the Bowen Property

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The Bowen Property: Motivation and History

The Bowen property

A Fuchsian group G has the Bowen property if the limit set of *every* quasiconformal deformation of G is either a round circle or has Hausdorff dimension *strictly* greater than 1.

A dichotomy, established piece by piece:

- **Bowen:** holds for all *cocompact* Fuchsian groups ("Hausdorff dimension of quasicircles").
- **Sullivan:** extended it to *cofinite* groups.
- **Astala–Zinsmeister:** false for *convergence-type* groups of the first kind.

The only remaining case

Infinitely generated *divergence-type* groups. **This paper:** they *all* have the Bowen property.

Key Definitions

- **Kleinian group:** a discrete group of isometries of the hyperbolic three space $(\mathbb{H}^3, \mathbb{B}^3)$.
- **Fuchsian group** G**:** a Kleinian group fixing the unit disk $\mathbb{D} \subset \mathbb{R}^2 = \partial\mathbb{H}^3$.
- **Deformation:** a conformal map $\Phi : \mathbb{D} \to \Omega$ with $\Phi \circ g \circ \Phi^{-1}$ being a Möbius transform for every $g \in G$; it is **quasi-Fuchsian** if Φ extends quasiconformally to the plane.
- **Divergence- and convergence-type:** G is *divergence-type* if

$$\sum_{g \in G} e^{-\rho(0, g(0))} = \infty \qquad (\rho = \text{hyperbolic metric}),$$

  and *convergence-type* otherwise.
- **Critical (Poincaré) exponent:**

$$\delta(G) = \inf\left\{ s : \sum_{g \in G} e^{-s\,\rho(0, g(0))} < \infty \right\}.$$

- **First kind:** the limit set is the whole circle.

Standard inclusions: cocompact $\subset$ cofinite $\subset$ divergence-type $\subset$ first kind.

Main Results

Theorem 1

Let G be a divergence-type Fuchsian group and $G' = \Phi \circ G \circ \Phi^{-1}$ a deformation. Then *either* $\partial\Omega$ is a circle *or* $\delta(G') > 1$.

Combined with $\delta(G) \leq \dim(\Lambda)$ for nonelementary Kleinian groups (Bishop–Jones), Theorem 1 yields:

every divergence group has the Bowen property.

Astala–Zinsmeister proved the converse (Bowen $\Rightarrow$ divergence), hence:

Corollary 2

A Fuchsian group is divergence-type *if and only if* it has the Bowen property.

Proof Strategy: Reduce to a Tree of Orbit Points

Key new idea: use *three-dimensional* hyperbolic geometry to estimate the expansion of planar conformal maps.

Reformulation (Corollary 4)

$$\delta(G') = \inf\left\{ s : \sum_{g\in G} |\Phi'(g(0))|^{s} (1-|g(0)|^2)^s < \infty \right\}.$$

Goal: find $s > 1$ making this series diverge. It suffices to build a tree $\mathcal{C} = \bigcup_n \mathcal{C}_n$ of orbit points, $\mathcal{C}_0 = \{0\}$, where each z has finitely many children $\mathcal{C}(z)$ with

(1) **bounded steps:** $\sup_z \sup_{w\in\mathcal{C}(z)} \rho(w,z) \le D_0$;

(2) **growth:** $\sum_{w\in\mathcal{C}(z)} |\Phi'(w)|(1-|w|^2) \;\ge\; M_0\, |\Phi'(z)|(1-|z|^2), \quad M_0 > 1.$

For a suitable $s = s(D_0, M_0) > 1$ the level sums stay ≥ 1, forcing divergence, so $\delta(G') > 1$.

Two Important Lemmas

Lemma 5 (Growth near orbits)

Suppose G is a divergence group, Q is a Carleson square, and $z = g(0) \in T(Q)$ for some $g \in G$. Suppose Φ is a conformal deformation which is *not* a Möbius transformation. Then for any $M > 0$ there is a collection of disjoint subsquares $\{Q_j\} \subset Q$ and points $z_j \in T(Q_j)$ (*not* necessarily orbit points) such that:

- $\sum_j |\Phi'(z_j)|\,(1-|z_j|^2) \;\geq\; M\,|\Phi'(z)|\,(1-|z|^2)$,
- $1-|z_j|^2 \;\geq\; \delta\,(1-|z|^2)$ for some $\delta > 0$ depending only on G, Φ, and M.

Proof idea. Via normal families and Pommerenke's theorem: the limit set of a quasi-Fuchsian deformation of a divergence group is a circle or has tangents almost nowhere. Key tool: the hyperbolic Schwarzian $S_H(\Phi)(z) = |S(\Phi)(z)|(1-|z|^2)^2$ is constant on orbits of G, giving uniform control across the orbit.

Two Important Lemmas (continued)

Lemma 6 (Uniform renormalization) new contribution

Suppose G is a divergence-type Fuchsian group, $G' = \Phi \circ G \circ \Phi^{-1}$ is a quasi-Fuchsian deformation, Q is a Carleson square, and $z \in T(Q)$. Then there is a collection of disjoint Carleson subsquares $\{Q_j\} \subset Q$ and points $z_j \in T(Q_j) \cap G(0)$ such that:

- $\sum_j |\Phi'(z_j)|\,(1 - |z_j|^2) \;\geq\; \varepsilon\,|\Phi'(z)|\,(1 - |z|^2)$, where ε depends on G and Φ but *not* on z;
- $\inf_j (1 - |z_j|^2) \;\geq\; \eta\,(1 - |z|^2)$, where $\eta > 0$ depends only on G and $\rho(z, G(0))$.

Proof idea. Take the convex hull of $\partial\Omega$ in $\mathbb{B}^3$; its boundary S facing Ω is isometric to $\mathbb{H}^2$ (Thurston). Sullivan's theorem gives a K-qc biLipschitz map $\sigma : \Omega \to S$. Set $\Psi = \iota \circ \sigma \circ \Phi$: a qc self-map of $\mathbb{D}$ conjugating G to $H = \Psi \circ G \circ \Psi^{-1}$ (divergence-type by Pfluger). Nontangential density of $H(0)$ on $\partial\mathbb{D}$ then supplies the orbit points $z_j \in G(0)$.