Extremal Length and Duality

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Classical Modulus of a Curve Family

Let Γ be a family of rectifiable curves, i.e. curves of finite variation, on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let dA denote the spherical area element on the Riemann sphere: $dA(z) = \frac{4\,dx\,dy}{(1+|z|^2)^2}$.

Admissible Density

A Borel function

$$\rho : \widehat{\mathbb{C}} \to [0, \infty]$$

is called admissible for Γ if

$$\int_\gamma \rho\, ds \geq 1 \qquad \text{for all } \gamma \in \Gamma.$$

Conformal Modulus

$$\mathrm{mod}(\Gamma) = \inf_\rho \int_{\widehat{\mathbb{C}}} \rho^2\, dA,$$

where the infimum is taken over all admissible densities ρ.

Model Example: A Rectangle

Let

$$R = [0, a] \times [0, b].$$

Denote by Γ_1 the family of curves joining the vertical sides, and by Γ_2 the family of curves joining the horizontal sides.

Grötzsch's Formulas

$$\operatorname{mod}(\Gamma_1) = \frac{b}{a}, \qquad \operatorname{mod}(\Gamma_2) = \frac{a}{b}.$$

Therefore,

$$\boxed{\operatorname{mod}(\Gamma_1)\operatorname{mod}(\Gamma_2) = 1.}$$

This is the basic duality around which the entire talk will be built.

Upper Bound for $\mathrm{mod}(\Gamma_1)$

Let

$$R = [0, a] \times [0, b], \qquad \Gamma_1 = \{\gamma : \gamma \text{ joins the vertical sides}\}.$$

Take the constant density $\rho_0 \equiv \frac{1}{a}$.
For any $\gamma \in \Gamma_1$, its horizontal projection has length at least a. Hence

$$\int_\gamma \rho_0 \, ds = \frac{1}{a}\ell(\gamma) \geq \frac{1}{a} \cdot a = 1.$$

Thus ρ_0 is admissible. Therefore

$$\mathrm{mod}(\Gamma_1) \leq \int_R \rho_0^2 \, dA = \int_R \frac{1}{a^2} \, dA = \frac{ab}{a^2} = \frac{b}{a}.$$

$$\boxed{\mathrm{mod}(\Gamma_1) \leq \frac{b}{a}.}$$

Lower Bound for $\mathrm{mod}(\Gamma_1)$

Let ρ be any admissible density for Γ_1.
For every $y \in [0, b]$, the horizontal segment

$$\gamma_y(t) = (t, y), \qquad 0 \le t \le a,$$

belongs to Γ_1. Therefore $\int_0^a \rho(x, y)\, dx \ge 1$ for all y.
Integrating with respect to y, we obtain

$$b \le \int_0^b \int_0^a \rho(x, y)\, dx\, dy.$$

By the Cauchy–Schwarz inequality,

$$\left(\int_R \rho\, dA \right)^2 \le \left(\int_R \rho^2\, dA \right) \left(\int_R 1\, dA \right) = ab \int_R \rho^2\, dA.$$

Hence

$$b^2 \le ab \int_R \rho^2\, dA \quad \Longrightarrow \quad \boxed{\mathrm{mod}(\Gamma_1) \ge \frac{b}{a}.}$$

Conformal Invariance of the Modulus

Theorem

Let $\Omega,\ \Omega' \subset \widehat{\mathbb{C}}$ be domains. Let $f : \Omega \to \Omega'$ be a conformal homeomorphism, and let Γ be a family of rectifiable curves in Ω. Then

$$\boxed{\operatorname{mod}(f\Gamma) = \operatorname{mod}(\Gamma).}$$

Conformality gives

$$|(f \circ \gamma)'(t)| = |f'(\gamma(t))|\,|\gamma'(t)|.$$

A density ρ on Ω is transformed into a density ρ' on Ω' by the rule

$$\rho'(f(z)) = \frac{\rho(z)}{|f'(z)|}.$$

Conversely,

$$\rho(z) = \rho'(f(z))|f'(z)|.$$

Conformal Invariance of the Modulus

Suppose that ρ and ρ' are related by

$$\rho'(f(z)) = \frac{\rho(z)}{|f'(z)|}.$$

Then, for every $\gamma \in \Gamma$,

$$\int_{f\circ\gamma} \rho'\, ds = \int_\alpha^\beta \rho'(f(\gamma(t)))|(f\circ\gamma)'(t)|\, dt$$
$$= \int_\alpha^\beta \frac{\rho(\gamma(t))}{|f'(\gamma(t))|}|f'(\gamma(t))|\,|\gamma'(t)|\, dt = \int_\gamma \rho\, ds.$$

Thus admissibility is preserved in both directions.
Moreover,

$$\int_{\Omega'} (\rho')^2\, dA = \int_\Omega \left(\frac{\rho}{|f'|}\right)^2 |f'|^2\, dA = \int_\Omega \rho^2\, dA.$$

Consequently,

$$\boxed{\mathrm{mod}(f\Gamma) = \mathrm{mod}(\Gamma).}$$

Definition

A topological rectangle is a domain Q in $\widehat{\mathbb{C}}$, homeomorphic to a square, with four distinguished boundary arcs J_1, J_2, J_3, J_4 in cyclic order.

Let

$$\Gamma_1 = \{\text{curves joining } J_1 \text{ and } J_3\},$$

$$\Gamma_2 = \{\text{curves joining } J_2 \text{ and } J_4\}.$$

By the Riemann mapping theorem, there exists a conformal map

$$\varphi : Q \to [0, a] \times [0, b],$$

which maps the boundary arcs J_i to the sides of the Euclidean rectangle.
Since the modulus is conformally invariant, the moduli of the curve families in Q coincide with the moduli of their images in the rectangle. Hence

$$\boxed{\operatorname{mod}(\Gamma_1)\operatorname{mod}(\Gamma_2) = 1.}$$

An Example of Computing the Modulus

Consider a concentric annulus on the Riemann sphere:

$$A(r,R) = \mathbb{D}(0,R) \setminus \overline{\mathbb{D}(0,r)}, \qquad 0 < r < R.$$

Let $\Gamma_{r,R}$ be the family of curves in $A(r,R)$ joining the two boundary components:

$$\{|z| = r\} \quad \text{and} \quad \{|z| = R\}.$$

Then

$$\boxed{\operatorname{mod}(\Gamma_{r,R}) = 2\pi \left(\log \frac{R}{r}\right)^{-1}.}$$

Annulus: Upper Bound

Take the radial density

$$\rho_0(z) = \frac{1}{|z| \log(R/r)}.$$

Let $\gamma \in \Gamma_{r,R}$. Then $|\gamma|$ goes from radius r to radius R. Therefore

$$\int_\gamma \frac{1}{|z|}\, ds \geq \int_r^R \frac{dt}{t} = \log \frac{R}{r}.$$

Consequently,

$$\int_\gamma \rho_0 \, ds \geq \frac{1}{\log(R/r)} \log \frac{R}{r} = 1.$$

Thus ρ_0 is admissible. Hence

$$\operatorname{mod}(\Gamma_{r,R}) \leq \int_{A(r,R)} \rho_0^2 \, dA.$$

Annulus: Energy of the Extremal Density

Compute the energy in polar coordinates:

$$z = se^{i\theta}, \qquad dA = s\,ds\,d\theta.$$

$$\begin{aligned}
\int_{A(r,R)} \rho_0^2\,dA &= \int_0^{2\pi}\int_r^R \frac{1}{s^2\log^2(R/r)}\,s\,ds\,d\theta \\
&= \frac{2\pi}{\log^2(R/r)}\int_r^R \frac{ds}{s} \\
&= \frac{2\pi}{\log^2(R/r)}\log\frac{R}{r} = 2\pi\left(\log\frac{R}{r}\right)^{-1}.
\end{aligned}$$

Thus

$$\boxed{\operatorname{mod}(\Gamma_{r,R}) \le 2\pi\left(\log\frac{R}{r}\right)^{-1}.}$$

The lower bound is proved in the same way as in the case of the rectangle.

Metric Sphere

Definition

A metric sphere is a metric space

$$X = (X, d_X),$$

which is homeomorphic, as a topological space, to the Riemann sphere:

$$X \cong \widehat{\mathbb{C}}.$$

Denote by $\mathcal{H}^2(\cdot)$ the two-dimensional Hausdorff measure.

Definition

A metric sphere X is called Ahlfors 2-regular if there exists $C \geq 1$ such that, for every $x \in X$ and every $0 < r < \operatorname{diam} X$,

$$C^{-1} r^2 \leq \mathcal{H}^2(B(x, r)) \leq C r^2.$$

Linearly Locally Connected Spaces

LLC

A space X is called linearly locally connected if there exists $\lambda \geq 1$ such that:

LLC1. If

$$y, z \in B(x, r),$$

then y and z can be joined by a path inside

$$B(x, \lambda r).$$

LLC2. If

$$y, z \in X \setminus B(x, r),$$

then y and z can be joined by a path inside

$$X \setminus B(x, r/\lambda).$$

Quasisymmetric Homeomorphism

Let (X, d_X) and (Y, d_Y) be metric spaces.

Definition

A homeomorphism $f : X \to Y$ is called quasisymmetric if there exists a homeomorphism

$$\eta : [0, \infty) \to [0, \infty)$$

such that, for any distinct points $x, a, b \in X$,

$$\frac{d_Y(f(x), f(a))}{d_Y(f(x), f(b))} \leq \eta\left(\frac{d_X(x, a)}{d_X(x, b)}\right).$$

Bonk–Kleiner Theorem

Classical Uniformization Theorem for the Sphere

Let S be a Riemann surface which is topologically homeomorphic to the sphere: $S \cong \widehat{\mathbb{C}}$.
Then there exists a conformal homeomorphism $\varphi : \widehat{\mathbb{C}} \to S$.

Classical uniformization requires a complex structure on S. We want to obtain its metric analogue.

Bonk–Kleiner Theorem

Let X be an Ahlfors 2-regular metric sphere. Then there exists a quasisymmetric homeomorphism

$$f : \widehat{\mathbb{C}} \to X$$

if and only if X is LLC.

Let Z be a λ-LLC metric space, and let $f : Z \to Y$ be an η-quasisymmetric homeomorphism.

Claim

Then Y is $\eta(\lambda)$-LLC.

Let $y, z \in B_Y(x, r)$. Denote the preimages by

$$\tilde{x} = f^{-1}(x), \qquad \tilde{y} = f^{-1}(y), \qquad \tilde{z} = f^{-1}(z).$$

Set

$$R = \max\{d_Z(\tilde{x}, \tilde{y}), d_Z(\tilde{x}, \tilde{z})\}.$$

Choose $\tilde{a} \in \{\tilde{y}, \tilde{z}\}$ such that

$$d_Z(\tilde{x}, \tilde{a}) = R.$$

Since Z is λ-LLC, the points $\tilde{y}, \tilde{z}$ can be joined by a path

$$E \subset B_Z(\tilde{x}, \lambda R).$$

Now, for every $u \in E$,

$$\frac{d_Y(x, f(u))}{d_Y(x, f(\tilde{a}))} \le \eta\left(\frac{d_Z(\tilde{x}, u)}{d_Z(\tilde{x}, \tilde{a})}\right) \le \eta(\lambda).$$

Since

$$\tilde{a} \in \{\tilde{y}, \tilde{z}\},$$

we have

$$f(\tilde{a}) \in \{y, z\} \subset B_Y(x, r).$$

Hence

$$d_Y(x, f(\tilde{a})) < r.$$

From the preceding estimate,

$$d_Y(x, f(u)) \leq \eta(\lambda) d_Y(x, f(\tilde{a})) < \eta(\lambda) r.$$

Therefore

$$f(E) \subset B_Y(x, \eta(\lambda) r).$$

By the continuity of f, $f(E)$ is also a path.
Thus any two points $y, z \in B_Y(x, r)$ can be joined by a path inside

$$B_Y(x, \eta(\lambda) r).$$

This is precisely LLC1 for Y. LLC2 is proved in a similar way.

Modulus on a Metric Surface

Definition

A metric surface is a metric space $X = (X, d_X)$, such that every point has a neighborhood homeomorphic to the open disk $\mathbb{D} \subset \mathbb{R}^2$.

Definition

The modulus of a curve family Γ on a metric surface X is

$$\operatorname{mod}_X(\Gamma) = \inf_{\rho} \int_X \rho^2 \, d\mathcal{H}^2_X,$$

where the infimum is taken over all admissible densities ρ.

An admissible density is defined in the same way as in the complex case:

$$\int_\gamma \rho \, ds_X = \int_0^{\ell_X(\gamma)} \rho(\gamma(x)) \, dx \geq 1 \quad \text{for all } \gamma \in \Gamma.$$

Quasiconformality via Modulus

Let X, Y be metric surfaces, and let $f : X \to Y$ be a homeomorphism.

Modular Definition

f is called K-quasiconformal if, for every curve family Γ in X,

$$\boxed{\frac{1}{K}\,\mathrm{mod}_X(\Gamma) \le \mathrm{mod}_Y(f\Gamma) \le K\,\mathrm{mod}_X(\Gamma).}$$

Let now $\Omega, \Omega' \subset \mathbb{C}$ be planar domains, and let $f : \Omega \to \Omega'$ be a smooth homeomorphism.
Consider the analytic condition of quasiconformality:

$$\|Df_z\|^2 \le KJ_f(z), \qquad \text{where } J_f(z) = |\det Df_z|.$$

Fact

In the classical planar case, the modulus definition of quasiconformality is equivalent to the analytic condition

$$\|Df_z\|^2 \le KJ_f(z) \quad \text{almost everywhere.}$$

Proof of One Direction

We prove the implication:

$$\boxed{\|Df_z\|^2 \le KJ_f(z) \implies \text{modular } K\text{-QC.}}$$

Let ρ' be admissible for $f\Gamma$. Define a density on Ω by

$$\rho(z) = \rho'(f(z))\|Df_z\|.$$

For every $\gamma \in \Gamma$,

$$|(f \circ \gamma)'(t)| = |Df_{\gamma(t)}\gamma'(t)| \le \|Df_{\gamma(t)}\|\,|\gamma'(t)|.$$

Therefore,

$$\begin{aligned}\int_\gamma \rho\,ds &= \int \rho'(f(\gamma(t)))\|Df_{\gamma(t)}\|\,|\gamma'(t)|\,dt \\ &\ge \int \rho'(f(\gamma(t)))|(f\circ\gamma)'(t)|\,dt = \int_{f\circ\gamma} \rho'\,ds \ge 1.\end{aligned}$$

Hence ρ is admissible for Γ.

Energy Estimate

We use $\rho(z) = \rho'(f(z))\|Df_z\|$ and the analytic condition $\|Df_z\|^2 \leq KJ_f(z)$.
Then

$$\int_\Omega \rho^2\,dA = \int_\Omega \rho'(f(z))^2\|Df_z\|^2\,dA(z)$$
$$\leq K\int_\Omega \rho'(f(z))^2 J_f(z)\,dA(z).$$

By the change of variables formula,

$$\int_\Omega \rho'(f(z))^2 J_f(z)\,dA(z) = \int_{\Omega'} (\rho')^2\,dA.$$

Thus,

$$\int_\Omega \rho^2\,dA \leq K\int_{\Omega'} (\rho')^2\,dA.$$

Taking the infimum over all admissible ρ', we obtain

$$\boxed{\operatorname{mod}_X(\Gamma) \leq K\operatorname{mod}_Y(f\Gamma).}$$

The Second Inequality and the Conclusion

To obtain the second estimate

$$\boxed{\operatorname{mod}_Y(f\Gamma) \leq K \operatorname{mod}_X(\Gamma),}$$

we apply the same argument to the inverse map

$$f^{-1} : Y \to X.$$

Thus,

$$\boxed{\frac{1}{K} \operatorname{mod}_X(\Gamma) \leq \operatorname{mod}_Y(f\Gamma) \leq K \operatorname{mod}_X(\Gamma).}$$

The Reverse Implication in the Smooth Case

Assume that f is K-quasiconformal in the modular sense:

$$\frac{1}{K}\,\mathrm{mod}(\Gamma) \leq \mathrm{mod}(f\Gamma) \leq K\,\mathrm{mod}(\Gamma)$$

for every family of curves Γ in Ω.

Goal

To prove the classical analytic condition:

$$\boxed{\|Df_z\|^2 \leq KJ_f(z)}$$

at every point $z \in \Omega$.

Singular Values of the Derivative

Fix a point $z_0 \in \Omega$.
The derivative $Df_{z_0} : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map.
Denote its singular values by

$$s_1 \geq s_2 > 0.$$

Then

$$\|Df_{z_0}\| = s_1, \qquad J_f(z_0) = s_1 s_2.$$

Therefore, the condition

$$\|Df_{z_0}\|^2 \leq K J_f(z_0)$$

is equivalent to

$$s_1^2 \leq K s_1 s_2,$$

that is,

$$\boxed{\frac{s_1}{s_2} \leq K.}$$

It is enough to prove precisely this inequality.

A Small Rectangle Around the Point

Choose coordinates so that

$$Df_{z_0}(x, y) = (s_1 x, s_2 y).$$

Consider the small rectangle

$$Q_\varepsilon = z_0 + \varepsilon([-a, a] \times [-b, b]).$$

Let Γ_ε be the family of curves in Q_ε joining the vertical sides.
The modulus of a rectangle is invariant under scaling; hence

$$\operatorname{mod}(\Gamma_\varepsilon) = \frac{b}{a}.$$

By modular K-quasiconformality,

$$\operatorname{mod}(f\Gamma_\varepsilon) \geq \frac{1}{K} \operatorname{mod}(\Gamma_\varepsilon) = \frac{1}{K} \frac{b}{a}.$$

At a Small Scale, f Looks Like Df_{z_0}

Define the normalized maps

$$F_\varepsilon(u) = \frac{f(z_0 + \varepsilon u) - f(z_0)}{\varepsilon}.$$

Since f is differentiable at z_0,

$$F_\varepsilon(u) \to Df_{\mathbf{z_0}}(u)$$

uniformly on the fixed rectangle

$$Q = [-a, a] \times [-b, b].$$

Thus, on an infinitesimal scale, f behaves like the linear map $A = Df_{\mathbf{z_0}}$.
We use the standard fact that modulus is stable under C^1-convergence:

$$\operatorname{mod}(f\Gamma_\varepsilon) \longrightarrow \operatorname{mod}(A\Gamma),$$

where Γ is the family of curves in Q joining the vertical sides.

The Modulus of the Linear Image

The linear map $A(x, y) = (s_1 x, s_2 y)$ sends the rectangle

$$[-a, a] \times [-b, b]$$

to the rectangle

$$[-s_1 a, s_1 a] \times [-s_2 b, s_2 b].$$

The family $A\Gamma$ joins the vertical sides of the new rectangle.
By the formula for a rectangle,

$$\operatorname{mod}(A\Gamma) = \frac{\text{height}}{\text{width}} = \frac{2s_2 b}{2s_1 a} = \frac{s_2}{s_1}\frac{b}{a}.$$

On the other hand, the modular K-QC estimate gives

$$\operatorname{mod}(f\Gamma_\varepsilon) \geq \frac{1}{K}\frac{b}{a}.$$

Passing to the limit as $\varepsilon \to 0$, we obtain

$$\frac{s_2}{s_1}\frac{b}{a} \geq \frac{1}{K}\frac{b}{a}.$$

Obtaining the Analytic Condition

From the inequality

$$\frac{s_2}{s_1}\frac{b}{a} \geq \frac{1}{K}\frac{b}{a}$$

it follows that

$$\frac{s_2}{s_1} \geq \frac{1}{K}.$$

That is,

$$\boxed{\frac{s_1}{s_2} \leq K.}$$

Now recall that

$$\|Df_{z_0}\| = s_1, \qquad J_f(z_0) = s_1 s_2.$$

Therefore,

$$\|Df_{z_0}\|^2 = s_1^2 \leq K s_1 s_2 = K J_f(z_0).$$

Since z_0 was arbitrary,

$$\boxed{\|Df_z\|^2 \leq K J_f(z)}$$

for all $z \in \Omega$.

Why Modulus Is Stable Under C^1-Convergence

It is enough to use the following simple fact.

Lemma

Let $H : U \to V$ be an L-bi-Lipschitz homeomorphism:

$$L^{-1}|x-y| \le |H(x)-H(y)| \le L|x-y|.$$

Then for every family of curves Γ in U,

$$L^{-4}\,\mathrm{mod}(\Gamma) \le \mathrm{mod}(H\Gamma) \le L^{4}\,\mathrm{mod}(\Gamma).$$

Therefore, if $L \to 1$, then

$$\mathrm{mod}(H\Gamma) \to \mathrm{mod}(\Gamma).$$

Modulus is stable under small bi-Lipschitz perturbations.

Proof of the Bi-Lipschitz Lemma

We prove one side: $\operatorname{mod}(H\Gamma) \geq L^{-4} \operatorname{mod}(\Gamma)$.
Let ρ' be admissible for $H\Gamma$. Define a density on U by

$$\rho(x) = L\rho'(H(x)).$$

For every $\gamma \in \Gamma$, we have

$$\begin{aligned}\int_\gamma \rho\, ds &= \int_\gamma L\rho'(H(x))\, ds \\ &\geq \int_{H\circ\gamma} \rho'\, ds.\end{aligned}$$

Since ρ' is admissible for $H\Gamma$,

$$\int_{H\circ\gamma} \rho'\, ds \geq 1.$$

Hence,

$$\int_\gamma \rho\, ds \geq 1.$$

Thus ρ is admissible for Γ.

Proof of the Bi-Lipschitz Lemma: Energy

Now we estimate the energy:

$$\int_U \rho^2 \, dA = L^2 \int_U \rho'(H(x))^2 \, dA(x).$$

Since H^{-1} is L-Lipschitz, areas of preimages increase by at most a factor of L^2. Therefore,

$$\int_U \rho'(H(x))^2 \, dA(x) \leq L^2 \int_V (\rho')^2 \, dA.$$

Consequently,

$$\int_U \rho^2 \, dA \leq L^4 \int_V (\rho')^2 \, dA.$$

Taking the infimum over all admissible ρ', we get

$$\operatorname{mod}(\Gamma) \leq L^4 \operatorname{mod}(H\Gamma).$$

That is,

$$\boxed{\operatorname{mod}(H\Gamma) \geq L^{-4} \operatorname{mod}(\Gamma).}$$

The other side is proved by applying the same estimate to H^{-1}.

Application to C^1-Convergence

We have

$$F_\varepsilon(u) = \frac{f(z_0 + \varepsilon u) - f(z_0)}{\varepsilon}$$

and

$$F_\varepsilon \to A = Df_{z_0}$$

in C^1 on the fixed rectangle Q.
Consider

$$H_\varepsilon = F_\varepsilon \circ A^{-1}.$$

Then $H_\varepsilon \to \mathrm{id}$ in C^1 on $A(Q)$.
Therefore, for small ε, the map H_ε is L_ε-bi-Lipschitz, where

$$L_\varepsilon \to 1.$$

By the lemma,

$$L_\varepsilon^{-4}\,\mathrm{mod}(A\Gamma) \le \mathrm{mod}(F_\varepsilon\Gamma) \le L_\varepsilon^{4}\,\mathrm{mod}(A\Gamma).$$

Since $L_\varepsilon \to 1$, we obtain

$$\boxed{\mathrm{mod}(F_\varepsilon\Gamma) \to \mathrm{mod}(A\Gamma).}$$

Reciprocity Condition

Let X be a metric sphere, and let $Q \subset X$ be a topological rectangle with boundary arcs J_1, J_2, J_3, J_4 in cyclic order.
Define the dual families of curves:

$$\Gamma_1 = \{\text{curves in } Q \text{ joining } J_1 \text{ and } J_3\},$$

$$\Gamma_2 = \{\text{curves in } Q \text{ joining } J_2 \text{ and } J_4\}.$$

Reciprocity Condition

X satisfies the reciprocity condition if there exists $\kappa \geq 1$ such that, for every such Q,

$$\boxed{\kappa^{-1} \leq \operatorname{mod}_X(\Gamma_1)\operatorname{mod}_X(\Gamma_2) \leq \kappa.}$$

A QC Parametrization Implies Reciprocity

Let $f : \widehat{\mathbb{C}} \to X$ be a K-quasiconformal homeomorphism in the modular sense.
Take a topological rectangle $Q \subset X$. Then

$$Q' = f^{-1}(Q) \subset \widehat{\mathbb{C}}$$

is also a topological rectangle.
Let Γ_1', Γ_2' be the preimages of the families Γ_1, Γ_2. In the classical case,

$$\operatorname{mod}(\Gamma_1') \operatorname{mod}(\Gamma_2') = 1.$$

Quasiconformality gives

$$K^{-1} \operatorname{mod}(\Gamma_i') \le \operatorname{mod}_X(\Gamma_i) \le K \operatorname{mod}(\Gamma_i'), \qquad i = 1, 2.$$

Multiplying these inequalities, we obtain

$$\boxed{K^{-2} \le \operatorname{mod}_X(\Gamma_1) \operatorname{mod}_X(\Gamma_2) \le K^2.}$$

Thus X satisfies the reciprocity condition with $\kappa = K^2$.

Rajala's Theorem

We have proved the simple direction:

$$\text{QC parametrization} \implies \text{reciprocity.}$$

Rajala's Theorem

For a metric sphere X,

$$\boxed{X \text{ is reciprocal} \iff \exists \text{ a quasiconformal homeomorphism } \widehat{\mathbb{C}} \to X.}$$

Weakly Quasiconformal Maps

When there is no quasiconformal homeomorphism $\widehat{\mathbb{C}} \to X$, one considers a weaker parametrization.

Definition

A map $f : \widehat{\mathbb{C}} \to X$ is called weakly K-quasiconformal if

$$f \text{ is continuous and surjective, and } f^{-1}(x) \text{ is connected for every } x \in X,$$

and, for every family of curves Γ in $\widehat{\mathbb{C}}$,

$$\boxed{\operatorname{mod}(\Gamma) \leq K \operatorname{mod}_X(f\Gamma).}$$

Why a Weak Parametrization Is Needed

Suppose that in the standard sphere we collapse a connected compact set

$$E \subset \widehat{\mathbb{C}}$$

to a single point.
The result is a metric sphere X, but the natural map

$$\pi : \widehat{\mathbb{C}} \to X$$

is no longer a homeomorphism:

$$\pi(x) = \pi(y) \qquad \text{for all } x, y \in E.$$

However, it is monotone:

$$\pi^{-1}(p) = E$$

is connected. In this case, a weak uniformization exists.

Weak uniformization of metric spheres

Theorem of Ntalampekos–Romney; Meier–Wenger and Rajala

Let X be a metric sphere. Then there exists a weakly $\frac{4}{\pi}$ -quasiconformal map $f : \widehat{\mathbb{C}} \to X$. Moreover:

1. The constant $\frac{4}{\pi}$ is the smallest universal constant.
2. f is a quasiconformal homeomorphism if and only if X satisfies the reciprocity condition: $\kappa^{-1} \leq \mathrm{mod}_X(\Gamma_1)\,\mathrm{mod}_X(\Gamma_2) \leq \kappa$.
3. If X satisfies upper 2-regularity,
$$\mathcal{H}^2(B(x,r)) \leq Cr^2,$$
then f is a quasiconformal homeomorphism.
4. If, in addition, X is LLC, then f is a quasisymmetric homeomorphism.

Transition to Section 3

So far, the main object has been: $\Gamma = \{\text{a family of curves}\}$.
Let us generalize this notion and construct a modulus for families of k-dimensional surfaces.

k-dimensional surface

$\lambda \subset \mathbb{R}^n$ is a countably k-rectifiable set: almost all of λ is covered by countably many Lipschitz images of subsets of $\mathbb{R}^k$.

That is, there exists a countable collection of Lipschitz maps $\varphi_j : A_j \subset \mathbb{R}^k \to \mathbb{R}^n$ such that

$$\mathcal{H}^k\left(\lambda \setminus \bigcup_j \varphi_j(A_j)\right) = 0.$$

Define the integral over λ, for a function ρ on $\mathbb{R}^n$, by

$$\int_\lambda \rho = \int_\lambda \rho \, d\mathcal{H}^k.$$

Modulus of families of k-surfaces

Let Λ be a family of k-dimensional surfaces in $\mathbb{R}^n$.

Admissible density for surfaces

A Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called admissible for Λ if

$$\int_\lambda \rho \, d\mathcal{H}^k \geq 1 \qquad \text{for every } \lambda \in \Lambda.$$

(k, p)-modulus

We define the modulus for surfaces as follows:

$$\operatorname{mod}_{k,p}(\Lambda) = \inf_\rho \int_{\mathbb{R}^n} \rho^p \, dx.$$

For $k = 1$, this is the usual p-modulus of curves:

$$\operatorname{mod}_{1,p}(\Gamma) = \operatorname{mod}_p(\Gamma).$$

Condenser: curves versus hypersurfaces

Let $G \subset \mathbb{R}^n$ be a bounded domain, and let $C_0, C_1 \subset G$ be disjoint compact sets. We call such a distinguished triple a condenser $C(G, C_0, C_1)$. Define two families.

$$\Gamma = \{\text{curves in } G \text{ joining } C_0 \text{ and } C_1\}. \qquad \Lambda = \{(n-1)\text{-surfaces separating } C_0 \text{ and } C_1\}.$$

Theorem (Gehring–Ziemer formula)

For every condenser $C(G, C_0, C_1)$, the following exact duality holds:

$$\boxed{\left(\mathrm{mod}_{1,n}(\Gamma)\right)^{1/n} \left(\mathrm{mod}_{n-1,n/(n-1)}(\Lambda)\right)^{(n-1)/n} = 1.}$$

Let us prove this. Take a potential function $u : G \to [0,1]$, $u = 0$ on C_0, $u = 1$ on C_1.
Almost all level sets $\{u = t\}$, $0 < t < 1$, separate C_0 and C_1.
If τ is admissible for Λ, then

$$\int_{\{u=t\}} \tau \, d\mathcal{H}^{n-1} \geq 1.$$

Coarea formula and Hölder's inequality

Integrating over $t \in (0,1)$, we get

$$1 \leq \int_0^1 \int_{\{u=t\}} \tau \, d\mathcal{H}^{n-1} \, dt.$$

By the coarea formula,

$$\int_0^1 \int_{\{u=t\}} \tau \, d\mathcal{H}^{n-1} \, dt = \int_G \tau |\nabla u| \, dx.$$

By Hölder's inequality,

$$\int_G \tau |\nabla u| \, dx \leq \left(\int_G \tau^{n/(n-1)} dx \right)^{(n-1)/n} \left(\int_G |\nabla u|^n dx \right)^{1/n}.$$

Hence

$$1 \leq \|\tau\|_{n/(n-1)} \|\nabla u\|_n.$$

Taking the infimum over all admissible τ and u, we obtain one side of the duality.

What this inequality gives

We have proved

$$\boxed{\left(\mathrm{mod}_n(\Gamma)\right)^{1/n}\left(\mathrm{mod}_{n-1,n/(n-1)}(\Lambda)\right)^{(n-1)/n} \geq 1.}$$

The other side requires a more delicate analysis of extremal densities.

What happens in intermediate dimensions? For the pair of dimensions

$$1 \quad \text{and} \quad n-1$$

there is an exact duality.
The natural question is:

$$\boxed{\text{is there an analogue for } k\text{- and } (n-k)\text{-dimensional surfaces?}}$$

In dimension 2, the duality of curves almost completely controls conformal geometry. In higher dimensions, this is no longer the case.

Homological setting

Let

$$Q \cong I^n = I^k \times I^{n-k}.$$

Define two parts of the boundary:

$$A = \partial I^k \times I^{n-k},$$

$$B = I^k \times \partial I^{n-k}.$$

We now consider two families:

$$\Lambda_A = \{k\text{-dimensional relative cycles generating } H_k(Q, A)\},$$

$$\Lambda_B = \{(n-k)\text{-dimensional relative cycles generating } H_{n-k}(Q, B)\}.$$

Λ_A and Λ_B are dual through intersection of cycles.

This is a higher-dimensional analogue of two transverse families of curves in a rectangle.

Lohvansuu's theorem

Lohvansuu's theorem

In the homological setting described above, one has

$$\boxed{\left(\operatorname{mod}_{k,n/k}(\Lambda_A)\right)^{k/n}\left(\operatorname{mod}_{n-k,n/(n-k)}(\Lambda_B)\right)^{(n-k)/n}\leq 1.}$$

This is an analogue of one side of the classical duality.

Open question

Is there in fact always equality here?

$$\left(\operatorname{mod}_{k,n/k}(\Lambda_A)\right)^{k/n}\left(\operatorname{mod}_{n-k,n/(n-k)}(\Lambda_B)\right)^{(n-k)/n}\stackrel{?}{=} 1.$$

Conclusion of the talk

We have seen one central idea:

How conformal geometry appears through duality of moduli.

1. In dimension 2:

$$\mathrm{mod}(\Gamma_1)\,\mathrm{mod}(\Gamma_2) = 1.$$

2. For metric spheres:

$$\text{reciprocity} \iff \text{QC parametrization.}$$

3. For arbitrary metric spheres:

there exists a weak QC uniformization.

4. In dimension n:

$$\text{curves} \leftrightarrow \text{separating hypersurfaces.}$$

5. For intermediate dimensions:

duality becomes homological and remains open.

Thank you!