Mostow Rigidity made easier

Léo Thiébaud

May 2026

Table of contents

Hyperbolic geometry

Definition

Let $\delta > 0$, a geodesic triangle is said to be $\delta-$thin if each edges lies in a $\delta-$neighbourhood of the other two edges.

Definition

A metric space is hyperbolic if there exists a $\delta > 0$ such that every geodesic triangle is $\delta-$thin.

Hyperbolic manifold

Definition

A manifold is a topological space locally modelled on some euclidean space $\mathbb{R}^d$

Definition

A riemaniann manifold is a manifold equipped with a metric. i.e. the data of a scalar product on each tangent space.

This allows the measure of the length of a path, and thus the definition of a geodesic.

Hyperbolic space

Example

Exemple : Upper half $\mathbb{R}\times]0,+\infty[$ plane with the following metric :

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$$

Or more generally $\mathbb{H}^d = \mathbb{R}^{d-1}\times]0,+\infty[$ with the metric :

$$\mathrm{d}s^2 = \frac{\sum_{k=1}^{n} \mathrm{d}x_k^2}{x_n^2}$$

Other model

Example

An equivalent model is the unit open ball : $\mathbb{H}^d \simeq \mathbb{B}^d$ with the following metric :

$$\mathrm{d}s^2 = 4\frac{\sum_{k=1}^{n} \mathrm{d}x_k^2}{(1 - \|x\|^2)^2}$$

definition

A riemaniann manifold is hyperbolic if it is locally isometric to some hyperbolic space $\mathbb{H}^d$.

(Quasi-)Isometry

Definition

An isometry is a map $f : (X_1, d_1) \longrightarrow (X_2, d_2)$ satisfying :

$$\forall x, y \in X_1, d_2(f(x), f(y)) = d_1(x, y)$$

Definition

A bi-Lipschtitz (BL) map* is a map $f : (X_1, d_1) \longrightarrow (X_2, d_2)$ satisfying :

$$\forall x, y \in X_1, \frac{1}{A} d_1(x, y) \leq d_2(f(x), f(y)) \leq A d_1(x, y)$$

Statement of the theorem

Theorem (Mostow, 1968) :

Let M_1 and M_2 be two $3-$dimensional compact hyperbolic manifolds, assume there exists a BL map between M_1 and M_2. Then there exists an actual isometry between M_1 and M_2.

Sketch of proof I II III and IV

Step 1 : Reduction to the case of Hyperbolic space (quotient of nice lattice).

Step 2 : BL extend to the boundary.

Step 3 : The extension is more regular than expected.

Step 4 : This extension induces an isometry.

Uniformisation

Uniformisation theorem

Any hyperbolic $3-$manifold is isometric to the quotient of the hyperbolic space by a nice* subgroup of its isometry group.

*Nice = acting freely and properly discontinuously on the hyperbolic space

If $f : M_1 = \mathbb{H}/\Gamma_1 \longrightarrow M_2 = \mathbb{H}/\Gamma_2$, then there exists essentially a unique $\tilde{f} : \mathbb{H} \longrightarrow \mathbb{H}$ satisfying :

- The induced quotient map of $\tilde{f}$ coincide with f.
- equivariance with respect to Γ_1 and Γ_2. I.E for any $\gamma_1 \in \Gamma_1$, there exists $\gamma_2 \in \Gamma_2$ such that : $\tilde{f} \circ \gamma_1 = \gamma_2 \circ \tilde{f}$

Replace

$$M_1, M_2 \text{ compact } \leftrightsquigarrow \mathbb{H}$$

$$f \text{ quasi-isometry} \leftrightsquigarrow \tilde{f} \text{ quasi-isometry} + \text{equivariance w.r.t. } \Gamma_{1,2}$$

Gromov Boundary

Let M be a hyperbolic manifold, we define ∂M as follow :
Fix $O \in M$ an origin point, and consider the set of geodesic starting at O, i.e. isometric embedding $\phi : \mathbb{R}_{\geq 0} \hookrightarrow M$ with $\phi(0) = O$. *
We say that two rays ϕ and ψ are equivalent if they stay at bounded distance from each other, i.e. if there exists $K > 0$ such that :

$$\forall t \geq 0, d(\phi(t), \psi) \leq K$$

Definition

We define the visual boundary ∂M of M as the set of equivalent class of geodesic rays.

Propriété

$$\partial\mathbb{H} \simeq \mathbb{S}^2 \simeq \mathbb{C} \cup \infty$$

Any isometry $H : \mathbb{H} \longrightarrow \mathbb{H}$ induces naturally a map $h : \partial\mathbb{H} \longrightarrow \partial\mathbb{H}$.

Theorem

There is a natural one-to-one correspondance between $Isom(M)$ and $Conf(\partial M)$

Moreover if H is equivariant, then so is h

Example

In our case, $Isom(\mathbb{H}^3)$ and $Conf(\mathbb{C} \cup \infty)$ are both isomorphic to $PSL(2, \mathbb{C})$

Quasi-geodesics

Definition

A quasi-geodesic in M is a li-Lipschitz embedding from $\mathbb{R}$ into M, i.e. a map $\phi : \mathbb{R} \longrightarrow M$ such that :

$$\exists A > 0, \forall s, t \in \mathbb{R}, \frac{1}{A}|s-t| \leq d(\phi(s), \phi(t)) \leq A|s-t|$$

Euclidean vs hyperbolic

Picture Euclidean quasi geodesic vs hyperbolic

Morse lemma

Morse lemma ():

Let M be a hyperbolic metric space, there exists a constant $K > 0$ such that for any quasi-geodesic ϕ in M, there exists an actual geodesic L that is $K-$close to ϕ i.e. :

$$\forall t \in \mathbb{R}, d(\phi(t), L) \leq K$$

Key point 1

The Morse lemma allows us to replace geodesics by quasi-geodesics in the definition of the Gromov boundary, without changing anything.
Thus, any quasi-isometry acts naturally on the boundary, and we denote h this extension.

Proposition

There is an extension map $QIsom(\mathbb{H}^3) \longrightarrow Diff(\mathbb{C} \cup +\infty)$

Key point 2

Let h be a function such that $h(0) = 0$ and h is differentiable at 0. By definition, this means that $\lim_{n\to\infty} nh(\frac{u}{n}) = A(u)$ where A is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Notice that $f_n(x) = nx$ and $g_n(x) = \frac{1}{n}x$ are affine conformal transformation of the plane, and that $A(x) = \lim(f_n h g_n)(x)$.

Main Idea

If we can give a meaning to the convergence of the sequences (f_n) and (g_n) toward isometries, this would imply that h is conjugated by isometries to a linear map, which means h is itself an isometry.