

Quasiconformal mappings and geodesics in the hyperbolic plane

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Basic knowledge

Before the explanation of the paper, we should review basic concepts. So I will introduce some knowledge about quasiconformal mappings, hyperbolic plane and geodesics in it.

1. Quasiconformal mappings

First of all, I will introduce basic knowledge about quasiconformal mappings. Let $w = f(z)$ ($z = x + iy, w = u + iv$)

be a C_1 homeomorphism from one region to another.

As we all know, $dw = f_z dz + f_{\bar{z}} d\bar{z}$.

Basic knowledge

Then we find that

$$f_z = \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y), f_{\bar{z}} = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y).$$

And we can obtain that $|f_z|^2 - |f_{\bar{z}}|^2 = u_x v_y - u_y v_x = J$
which is the Jacobian.

The Jacobian is positive for sense preserving and negative for sense reversing mappings.

Basic knowledge

We consider only the sense preserving case. It now follows that

$$(|f_z| - |f_{\bar{z}}|)|dz| \leq |dw| \leq (|f_z| + |f_{\bar{z}}|)|dz|,$$

where both limits can be attained.

Then we conclude that the ratio of the major to the minor axis is

$$D_f = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \geq 1.$$

We call it the *dilatation* at the point z . We also define

$$df = \frac{|f_{\bar{z}}|}{|f_z|} < 1.$$

Basic knowledge

Formal definition

The mapping f is said to be quasiconformal if D_f is bounded. It is K -quasiconformal if $D_f \leq K$. The mapping is conformal when K is 1.

Conformal mapping is a mapping that preserves angles between curves at each point. For example:

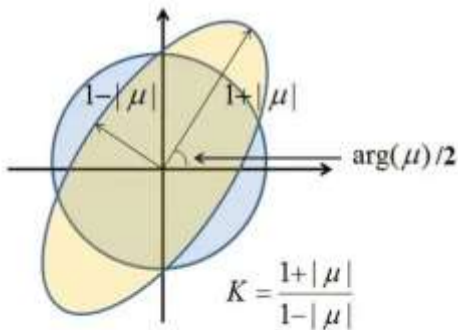
$$f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0.$$

we call it a fractional linear transformation, also known as a Mobius transformation.

Basic knowledge

Sketch map

By using quasiconformal mappings, we can map a circle to an ellipse, like this.



The ratio of the major axis to the minor axis of this ellipse equals K .

2. Hyperbolic plane

More knowledge about hyperbolic space can be found in [Ben].

Hyperbolic n -space, denoted by H^n , is the unique simply connected, complete Riemannian manifold of dimension n with constant sectional curvature -1 . We give different models for a real Riemannian n -manifold denoted by H^n , and these models will be by construction isometrically diffeomorphic to each other. They are *hyperboloid model*, *disc model* and *half-space model*. The disc model is the most famous one among them.

Basic knowledge

Disc model

We define π the restriction to I^n of the stereographic projection with respect to $(\mathbf{0}, \mathbf{0}, \dots, -\mathbf{1})$ of $\{\mathbf{x} \in \mathbb{R}^{n+1} : x_{n+1} > \mathbf{0}\}$ onto $\mathbb{R}^n \times \{\mathbf{0}\}$.

Let $I^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x} | \mathbf{x} \rangle_{(n,1)} = -\mathbf{1}, x_{n+1} > \mathbf{0}\}$, $\langle \mathbf{x} | \mathbf{y} \rangle_{(n,1)} = \sum_{i=1}^n x_i \cdot y_i - x_{n+1} \cdot y_{n+1}$.

Then we omit the last coordinate, so that \mathbb{R}^n is the range of π :

$$\pi(\mathbf{x}) = \frac{(x_1, \dots, x_n)}{1+x_{n+1}}.$$

Basic knowledge

We can prove that π is a diffeomorphism of I^n onto the open Euclidean unit ball D^n of R^n . And manifold D^n will be endowed with the pull-back metric with respect to π^{-1} .

When n equals 2, Hyperbolic n -space is *hyperbolic plane*. And we call D^2 *poincaré disk*, which is endowed with *hyperbolic metric*

$$g = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}$$

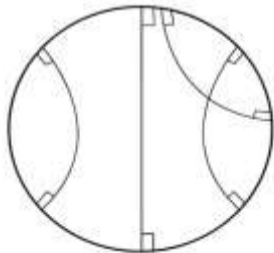
It is also a *Riemannian metric*.

Of course, the disk is a *Riemannian manifold*.

3. Geodesics

In short, a geodesic is the shortest path between two points in a given geometry. Geodesics in the hyperbolic plane, when mapped to the *poincaré disk*, become arcs or lines that are orthogonal to its boundary.

geodesics on the disk:



Basic knowledge

Half-space model

Let us consider the mapping:

$$i: D^2 \rightarrow R_+^2 \quad x \rightarrow 2 \frac{x+e_2}{\|x+e_2\|^2} - e_2$$

where $e_2 = (0, 1)$ and $R_+^2 = \{x \in R^2: x_2 > 0\}$ is the open upper half-plane. We check that the mapping is conformal and diffeomorphic. And there is a one-to-one correspondence between the two boundaries. It should be particularly noted that the boundary of the upper half-plane consists of the point at infinity and the real axis.

Basic knowledge

Under this mapping, the geodesics in the disk, which are arcs or diameters orthogonal to the boundary, are preserved as geodesics in the upper half-plane, which are vertical rays or semicircles, with the following correspondence.

Every disk geodesic that passes through the point $(0, 1)$, for example a diameter or arc emanating from that point, maps to a vertical ray in the upper half-plane.

Any other disk geodesic that does not pass through point $(0, 1)$ maps to a semicircle in the upper half-plane that joins two points on the real axis.

Abstract

After reviewing the basic concepts, we now move on to the explanation of the paper “*Quasiconformal mappings and geodesics in the hyperbolic plane*”, which is written by Ara Basmajian.

At first, we start with the paper’s abstract.

Abstract

In the paper we study the action of a quasiconformal mapping on the space of geodesics in H^2 , which is “quasi-isometry like” with respect to orthogonal distance. We use a known functional inequality to prove it.

I will introduce these concepts and prove the main theorem of the paper in detail.

Introduction

We want to show the relationship between deformations of complex structure and the underlying geometric quantities of the Riemann surface.

For example, we know that the length of the closed geodesic in a fixed free homotopy class of a closed curve is *quasi-invariant*.

In the context of quasiconformal mappings, a quantity is called quasi-invariant if its variation is controlled by the dilatation K of the mapping.

Introduction

The most typical example is the *extremal length* of a curve family:

If f is a K -quasiconformal mapping and Γ is a family of curves,

then $\frac{1}{K} \lambda(\Gamma) \leq \lambda(\Gamma^1) \leq K \lambda(\Gamma)$. $\lambda(\Gamma)$ is called the extremal length.

Suppose that all $\gamma \in \Gamma$ are contained in a region and let Γ^1 be the image of Γ under a K -quasiconformal mapping.

Moreover, Dirichlet integral of a C^1 function on a region is also quasi-invariant.

Introduction

This paper shows that a K -quasiconformal self-mapping of the hyperbolic plane, induces a *quasi-isometry* on the space of geodesics with respect to orthogonal distance between geodesics.

The mapping ϕ is *quasi-isometric*. Indeed, it satisfies

$$A^{-1}d[z_1, z_2] \leq d[\phi(z_1), \phi(z_2)] \leq A d[z_1, z_2]$$

with a constant A . And d is a non-Euclidean distance.

Introduction

There we stretch the definition of quasi-isometry to include *orthogonal distance*. We call the sequences of orthogonal distance *orthogonal spectrum*.

Let γ_0, γ_1 be geodesics in a Riemannian manifold (M, g) .

The orthogonal distance between them is $d(\gamma_0, \gamma_1) =$

$$\inf\{L(\sigma) \mid \sigma: [0, 1] \rightarrow M, \sigma(i) \in \gamma_i, \dot{\sigma}(i) \perp T_{\sigma(i)} \gamma_i, i = 0, 1\}$$

$\dot{\sigma}$ is the derivative of σ .

Introduction

Let f be a homeomorphism of \bar{H}^2 , and L is a geodesic in H^2 .

We use $[f(L)]$ to denote the geodesic having the same endpoints as $f(L)$. So the K -quasiconformal self-mapping of H^2 extends to a K -quasiconformal self-mapping of \hat{C} .

The following theorem we want to prove is the main result of the paper.

Theorem

THEOREM

Given $K \geq 1$ there exists a nonnegative constant $C=C(K)$ which goes to zero as K goes to 1, so that for any K -quasiconformal self-mapping of the hyperbolic plane and for any geodesics L_1 and L_2 in it,

we have $(1/K)d(L_1, L_2) - C \leq d([f(L_1)], [f(L_2)]) \leq Kd(L_1, L_2) + C$

We can let C be $\frac{1}{2} \cosh^{-1}(2^{-(K-1)^2} e^{6(K+1)^2 \sqrt{K-1}})$.

Theorem

In order to prove the theorem, we need a useful lemma.

Lemma: For any constants $c_2 \geq 1, K \geq 1,$

$$c_2 \cosh^K 2d + \sqrt{c_2^2 \cosh^{2K} 2d - 1} \leq (c_2 + \sqrt{c_2^2 - 1})e^{2Kd}, \text{ for } d \geq 0.$$

Proof: we make $x = \cosh(2d),$ we obtain that

$$c_2 x^K + \sqrt{c_2^2 x^{2K} - 1} \leq (c_1 + \sqrt{c_2^2 - 1})(x + \sqrt{x^2 - 1})^K, x \geq 1.$$

we find that when $x = 1,$ the left and right-hand sides of the inequality are equal. We use $f(x)$ and $g(x)$ to denote them.

So we just need to prove that $f \leq g.$

Theorem

Let $h(x) = \frac{x}{\sqrt{x^2-1}}$, and compute the derivative of f and g .

Obviously, $h(x)$ is a decreasing function. We have

$$\begin{aligned} \dot{f}(x) &= Kc_2x^{K-1}(1 + h(c_2x^K)) \\ &\leq K(c_2 + \sqrt{c_2^2 - 1})(x + \sqrt{x^2 - 1})^{K-1}(1 + h(c_2x^K)) \\ &\leq K(c_2 + \sqrt{c_2^2 - 1})(x + \sqrt{x^2 - 1})^{K-1}(1 + h(x)) \end{aligned}$$

The last expression is $\dot{g}(x)$. So we can prove that $\dot{f}(x) \leq \dot{g}(x)$ and $f(x) \leq g(x)$.

Theorem

Now we can prove the theorem.

Proof: Let L_1 and L_2 be two geodesics in the hyperbolic plane and let

L_1^1 and L_2^1 be geodesics in the image of the quasiconformal mapping.

We assume that the two geodesics are disjoint and not asymptotic.

Moreover, using the half-space model, we assume that L_1 and L_1^1 have their endpoints at 0 and ∞ , L_2 has its endpoints at 1 and $x > 1$, and L_2^1 has endpoints at 1 and x^1 . So $0, 1$ and ∞ are fixed points of the K -quasiconformal mapping.

Theorem

We will use the *cross-ratio* of four points on the extended complex plane. Let $a, b, c, d \in \widehat{\mathbb{C}}$, we define that

$$[a, b, c, d] = \frac{(a-c)(b-d)}{(a-d)(b-c)}.$$

So we set $s = |[\infty, \mathbf{0}, \mathbf{1}, x]| + |[\infty, \mathbf{0}, x, \mathbf{1}]| = \frac{x+1}{x-1}$,

and $s^1 = |[\infty, \mathbf{0}, \mathbf{1}, x^1]| + |[\infty, \mathbf{0}, x^1, \mathbf{1}]| = \frac{x^1+1}{x^1-1}$.

According to a functional inequality from [A-V-V], we have the inequality $s^1 \leq c_2 s^K$, $c_2 = \cosh(2C)$.

Theorem

We use d to denote the orthogonal distance from L_1 to L_2 and use d^1 to denote the distance from L_1^1 to L_2^1 .

A classical conclusion is $x = \coth^2(d)$.

So a simple computation shows that $s = \cosh(2d)$.

Using the preceding inequality, we have

$$\cosh(2d^1) \leq c_2 \cosh^K(2d)$$

Taking the inverse hyperbolic cosine of both sides, then

$$2d^1 \leq \log(c_2 \cosh^K 2d + \sqrt{c_2^2 \cosh^{2K} 2d - 1})$$

Theorem

We note that the lemma we have proved is useful there.

So we obtain that $2d^1 \leq \log(c_2 + \sqrt{c_2^2 - 1}) + 2Kd$,

$$\text{we have } d^1 \leq \frac{1}{2} \log \left(c_2 + \sqrt{c_2^2 - 1} \right) + Kd = \frac{1}{2} \cosh^{-1}(c_2) \\ + Kd = C + Kd.$$

We have proved the right-hand side of the inequality in the theorem. The left-hand side of the inequality holds in a similar way.

Others

We will show that the action of a quasiconformal mapping on the space of geodesics is not bilipschitz. That is to say, orthogonal distance is not quasi-invariant.

Consider the mapping $f_\rho: z \rightarrow |z|^{\rho-1}z$, $\rho \geq 1$. The mapping is an orientation preserving diffeomorphism of the extended complex plane and keeps the upper half-plane invariant.

$$\text{We compute the dilatation, } D_f(z) = \begin{cases} \rho, & \text{for } \rho \geq 1 \\ \frac{1}{\rho}, & \text{otherwise} \end{cases}$$

Others

Let d be the orthogonal distance between two geodesics which join the points $0, \infty$ and points $1, x(x \geq 1)$ respectively.

Due to $x = \coth^2 d$, we can conclude that

$$d = \frac{1}{2} \log \left(\frac{\sqrt{x}+1}{\sqrt{x}-1} \right)$$

which is asymptotic to $\frac{1}{\sqrt{x}}$ as x goes to infinity.

Similarly, we can also compute the orthogonal distance d^1 between geodesics $[f_\rho(L)]$ and $[f_\rho(L_x)]$.

Others

The distance is $\frac{1}{2} \log\left(\frac{\sqrt{x}^\rho + 1}{\sqrt{x}^\rho - 1}\right)$. We can find that $\frac{d^1}{d}$ for $\rho > 1$ will

be asymptotic to $\frac{\sqrt{x}}{\sqrt{x}^\rho}$ which goes to zero as x goes to infinity. So we prove that the action of the mapping on the geodesics is not bilipschitz.

But we can use a method called *Ahlfors-Beurling extension* to the the hyperbolic plane of the map f_ρ restricted to R . Then the extension will be bilipschitz with respect to the hyperbolic metric.

Others

The Ahlfors-Beurling extension is a standard method for converting a one-dimensional boundary function into a two-dimensional quasiconformal mapping. Its core lies in an integral averaging formula that systematically 'lifts' boundary values into the interior of the upper half-plane and preserve the quasiconformality of the mapping.

Its most important contribution is proving that determining whether a quasiconformal mapping exists is fully equivalent to determining whether its induced boundary function is quasisymmetric. Detailed content can be found in [Ah1].

References

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Thank you!