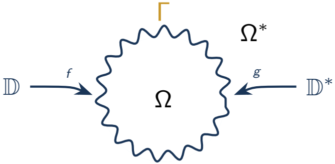
Conformal Welding and Koebe's Theorem

After C. J. Bishop, Annals of Mathematics 166 (2007)

Geometric Function Theory Student Conference



Outline

1. What is Conformal Welding?
2. The Main Theorem
3. Technical Background
4. Koebe's Theorem and Circle Chains
5. Extension of Partial Weldings
6. Quasisymmetric Maps are Weldings
7. Summary

Conformal Welding — Definition

Setup

Let $\mathbb{D}$ be an open unit disk on the Riemann sphere $\mathbb{S}^2$, $\mathbb{D}^* = \mathbb{S}^2 \setminus \overline{\mathbb{D}}$, $\mathbb{T} = \partial\mathbb{D}$.

Given a closed Jordan curve $\Gamma \subset \mathbb{S}^2$, let

$$f : \mathbb{D} \to \Omega, \qquad g : \mathbb{D}^* \to \Omega^*$$

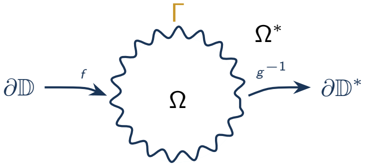
be conformal maps onto the bounded and unbounded complementary components.

Definition

The homeomorphism

$$h = g^{-1} \circ f : \mathbb{T} \longrightarrow \mathbb{T}$$

is called the **conformal welding** associated to Γ. We say Γ is a *welding curve* for h.



$h = g^{-1} \circ f : \mathbb{T} \to \mathbb{T}$

Conformal Welding — Definition

The correspondence between closed Jordan curves and homeomorphisms of the circle is **neither surjective nor injective even modulo Möbius transformations**.

Non-surjectivity (Remark 1)

Let K be the graph of $\sin(1/x)$ together with the segment $[-i, i]$. Then K divides $\mathbb{C}$ into two simply connected regions, and one can associate with K a homeomorphism $h\colon \mathbb{T} \to \mathbb{T}$ that is **not** a conformal welding.

Assuming the contrary we would be able to construct a conformal map

$$\mathbb{C} \setminus \{\text{segment}\} \to \mathbb{C} \setminus \{\text{point}\},$$

but such a map can not exist by Liouville's theorem.

Non-injectivity (Remark 2)

A Jordan curve Γ is *non-removable* if there is a homeomorphism $F\colon \mathbb{S}^2 \to \mathbb{S}^2$ conformal off Γ but not Möbius. Then Γ and $F(\Gamma)$ give the **same** h.

Simplest example: any curve of positive area. To prove this one can use the Beltrami equation with dilatation supported on Γ.

Generalized Conformal Weldings

Generalized welding (Hamilton 1991)

An orientation preserving homeomorphism $h\colon \mathbb{T} \to \mathbb{T}$ is a **generalized conformal welding** on $E \subset \mathbb{T}$ if there exist conformal $f\colon \mathbb{D} \to \Omega$, $g\colon \mathbb{D}^* \to \Omega^*$ (onto disjoint domains) such that f and g have radial limits on E and $h(E)$ respectively, satisfying $f(x) = g(h(x))$ for all $x \in E$.

Theorem 2

Any orientation-preserving homeomorphism h is a **generalized** conformal welding on $\mathbb{T} \setminus F$, where $F = F_1 \cup F_2$ and $\mathrm{cap}(F_1) = \mathrm{cap}(h(F_2)) = 0$.

Theorem 8 (Extension theorem)

Suppose $f\colon \mathbb{D} \to \Omega$, $g\colon \mathbb{D}^* \to \Omega^*$ are conformal maps onto disjoint Jordan domains and let $E = f^{-1}(\partial\Omega \cap \partial\Omega^*)$. Define $h = g^{-1} \circ f$ on E. Then h extends to a **full** conformal welding homeomorphism $\mathbb{T} \to \mathbb{T}$.

Bishop's Main Theorem

Theorem 1 — Main result

For *any* orientation-preserving homeomorphism $h\colon \mathbb{T} \to \mathbb{T}$ and any $\varepsilon > 0$, there exist a set $E \subset \mathbb{T}$ with $|E| + |h(E)| < \varepsilon$ and a conformal welding $H\colon \mathbb{T} \to \mathbb{T}$ such that $h(x) = H(x)$ for all $x \in \mathbb{T} \setminus E$.

Log-singular maps

A homeomorphism $h\colon \mathbb{T} \to \mathbb{T}$ is **log-singular** if $\exists$ Borel $E \subset \mathbb{T}$ with

$$\mathrm{cap}(E) = 0 \quad \text{and} \quad \mathrm{cap}\big(h(\mathbb{T} \setminus E)\big) = 0.$$

Intuition: h and its inverse "compress" log-full pieces of the circle into log-null pieces.

Theorem 1 can be proved for non-log-singular orientation preserving homeomorphisms by combining Theorem 2 and Theorem 8, though the final step from the result of Theorem 2 to application of Theorem 8 is not a trivial one.

Logarithmic Capacity

Definition

For a Borel set $E \subset \mathbb{T}$, define the *energy*

$$I(\mu) = \iint \log \tfrac{2}{|z-w|}\, d\mu(z)\, d\mu(w),$$

and the **logarithmic capacity**

$$\mathrm{cap}(E) = \frac{1}{\inf\{I(\mu) \colon \mu \in \mathrm{Prob}(E)\}}.$$

Physical intuition

Think of E as a conductor. Capacity measures how much "charge" E can hold:

- Countable sets have capacity 0.
- Any arc of $\mathbb{T}$ has positive capacity.
- A Cantor set can have *positive* capacity even when its Lebesgue measure is 0.

Extremal Length

Definition

For a family $\mathcal{P}$ of rectifiable paths in a domain Ω the **extremal length** of $\mathcal{P}$ is

$$\lambda(\mathcal{P}) = \sup_{\rho} \frac{\left(\inf_{\gamma \in \mathcal{P}} \int_{\gamma} \rho \, ds\right)^2}{\iint_{\Omega} \rho^2 \, dx \, dy},$$

where the supremum is taken over all non-negative Borel functions on Ω.
It is a *conformal invariant*.

Example: annulus

For the path family crossing the annulus $\{a < |z| < b\}$:

$$\lambda = 2\pi \log(b/a).$$

Wider annulus $\Longrightarrow$ larger extremal length $\Longrightarrow$ harder to cross.

Pfluger's Theorem (Lemma 12)

If $\mathcal{P}$ is the path family in $\mathbb{D}$ connecting a compact $K \subset r\mathbb{D}$, $0 < r \leq \frac{1}{3}$, to $E \subset \mathbb{T}$, then

$$\frac{\sqrt{r}}{1+r} \exp\left(-\frac{1}{\mathrm{cap}(E)}\right) \leq \exp\big(-\pi\,\lambda(\mathcal{P})\big).$$

Large extremal length $\iff$ small capacity.

Key lemma (Lemma 17)

If $f\colon \mathbb{D} \to \Omega$ is conformal, $\mathrm{dist}(f(0), \partial\Omega) = 1$, and for $R \geq 1$

$$E = \{x \in \mathbb{T} \colon |f(x)| \geq R\},$$

then $\mathrm{cap}(E) \leq CR^{-1/2}$ with C independent of Ω.

Geometric meaning: conformal maps can significantly "outstretch" boundary sets only if they have small capacity. This is a consequence of Pfluger's Theorem, as stronger outstretching results in greater extremal legth of certain path families.

Geometric construction: Theorem 4

Theorem 4

If h is orientation preserving homeomorphism and is *not* log-singular, there exist sequences of conformal maps $f_n \colon \mathbb{D} \to \Omega_n$ and $g_n \colon \mathbb{D}^* \to \Omega_n^*$ onto disjoint Jordan domains such that:

- $f_n(0) = 0$, $g_n(\infty) = \infty$;
- $\mathbb{S}^2 \setminus (\Omega_n \cup \Omega_n^*) \subset \{z \colon 1 \leq |z| \leq R\}$ with R independent of n;
- $|f_n(x) - g_n(h(x))| \to 0$ for all $x \in \mathbb{T} \setminus E$ for a countable set E.

Theorem 2 follows from Theorem 4:

1. Pass to subsequences such that $f_n \to f$, $g_n \to g$ uniformly on compact subsets.
2. The only way that $f(x) \neq g(h(x))$ off E is for $f(x) \neq \lim_n f_n(x)$ or $g(x) \neq \lim_n g_n(x)$ (or for the limits not to exist).
3. This discrepancy happens on at most one zero-capacity set for each "side". The argument uses extremal length estimates to derive an arbitrarily small upper bound on capacity of discrepancy subsets via Pfluger's theorem.

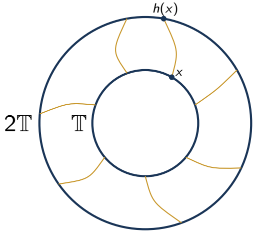
Core geometric idea behind Theorem 4 (Bishop, after Moore, Remark 3):

Moore decomposition

A decomposition of a $\mathbb{R}^2$ is a collection $\mathcal{C}$ of pairwise disjoint closed subsets whose union is the whole $\mathbb{R}^2$. It is upper-semicontinuous if each Hausdorff-convergent sequence of its elements converges to a subset of some element. If all elements of a decomposition $\mathcal{C}$ are continua which do not separate $\mathbb{R}^2$ and $\mathcal{C}$ is upper-semicontinuous, $\mathcal{C}$ is called a Moore decomposition of $\mathbb{R}^2$.

Moore's Theorem

If $\mathcal{C}$ is a Moore decomposition of $\mathbb{R}^2$, the quotient space is again homeomorphic to $\mathbb{R}^2$.



Foliation of A by arcs connecting x to $h(x)$

Bishop's insight (Remark 3)

Identifying each leaf to a point yields $\mathbb{R}^2$ again, and the annulus maps to a closed curve Γ. We cannot always collapse all leaves conformally, but Koebe's theorem lets us collapse *finitely many* at a time.

Koebe's Circle Domain Theorem (1908/1920)

Any finitely connected plane domain can be conformally mapped onto a domain bounded by **circles** and **points**.

How Bishop uses it:

1. Take n equally spaced points $\{x_k\} \subset \mathbb{T}$ and connect x_k to $h(x_k) \in 2\mathbb{T}$ by smooth arcs γ_k inside the annulus $A = \{1 < |z| < 2\}$.
2. Form $\Omega_{n,\varepsilon} = \mathbb{D} \cup 2\mathbb{D}^* \cup \bigcup_k (\varepsilon\text{-nbhd of } \gamma_k)$. This is finitely connected.
3. Apply Koebe to get $f_{n,\varepsilon}$ that maps complementary components to **disks**.
4. Ensure $f_{n,\varepsilon}(0) = 0$ and $f_{n,\varepsilon}(\infty) = \infty$.
5. Pass to limits $\varepsilon \to 0$ and $n \to \infty$ consequtively in this order.

Theorem 4 — Controlling the Chain Size

Why the chain stays in a bounded domain $R\mathbb{D}$ as $\varepsilon \to 0$ (Lemma 19):

- Capacity of arcs composing $\mathbb{T} \setminus \Omega_{n,\varepsilon}$ ($\mathbb{T} \setminus \Omega_{n,\varepsilon}$) with respect to 0 ($\infty$) is bounded below away from zero uniformly in ε.
- Then Lemma 17 implies that all disks must intersect both $\{|z| = M\}$ and $\{|z| = 2M\}$ for some M which does not depend on ε.
- This contradicts a simple fact

Lemma 20 (geometric key)

At most **six** disjoint disks can simultaneously intersect $\{|z| = 1\}$ and $\{|z| = 2\}$.

Disks must stay in a fixed annulus and hence will collapse to a **chain of tangent circles** for a properly chosen subsequence. Taking the limit of $f_{n,\varepsilon}$ we get f_n on $\mathbb{D}$ and g_n on $\mathbb{D}^*$.

Theorem 4 — Controlling the Chain Size

Why R stays bounded as $n \to \infty$ (Lemma 21):

- Suppose $R \to \infty$. Divide the chain disks into three groups: "inner", "outer", and "straddling". The straddling group has ≤ 6 disks by Lemma 20.
- The "inner" group preimages E_1 and "outer" group preimages E_2, together with preimages E_3 of at most six "stradders", cover all of $\mathbb{T}$.
- But $\operatorname{cap}(E_2 \cup E_3) + \operatorname{cap}(h(E_1)) \to 0$ by Lemma 17. This contradicts h being non-log-singular.

Consequence

At most $(R/\varepsilon)^2$ disks can have radius $\geq \varepsilon$. As $n \to \infty$, the remaining disks all shrink to points, so $f_n(x)$ and $g_n(h(x))$ both lie on the boundary of the *same* tiny disk — hence they converge to each other.

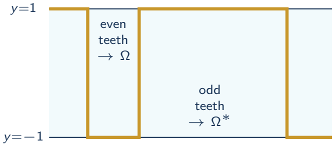
Taking a sequence $\varepsilon_n \to 0$ and diagonalizing we prove Theorem 4.

Theorem 8

Suppose $f\colon \mathbb{D} \to \Omega$ and $g\colon \mathbb{D}^* \to \Omega^*$ are conformal maps onto disjoint Jordan domains with $X = \partial\Omega \cap \partial\Omega^* \neq \emptyset$. Set $E = f^{-1}(X)$ and $h = g^{-1} \circ f$ on E. Then h extends to a **full conformal welding** $H\colon \mathbb{T} \to \mathbb{T}$.

Construction — the "oscillating curve" trick:

1. Each gap $U \in \{\text{components of } \mathbb{S}^2 \setminus (\Omega \cup \Omega^*)\}$ is a Jordan domain bounded by two analytic arcs (analiticity may be assumed modulo an auxiliary quasiconformal map).
2. Map U conformally to a horizontal strip $S = \{-1 < \operatorname{Im} z < 1\}$. The inverse of this map can be extended analytically a bit beyond two arcs bounding U into Ω and Ω^*.
3. Inside S, build an **oscillating curve**: alternating horizontal segments at $y = \pm 1$, connected by vertical segments — "teeth" pointing alternately up and down.



Oscillating curve

Theorem 8

Suppose $f\colon \mathbb{D} \to \Omega$ and $g\colon \mathbb{D}^* \to \Omega^*$ are conformal maps onto disjoint Jordan domains with $X = \partial\Omega \cap \partial\Omega^* \neq \emptyset$. Set $E = f^{-1}(X)$ and $h = g^{-1} \circ f$ on E. Then h extends to a **full conformal welding** $H\colon \mathbb{T} \to \mathbb{T}$.

Construction — the "oscillating curve" trick:

4. Attach the even teeth to Ω, the odd teeth to Ω^*. The result extends Ω and Ω^* to two sides of a Jordan curve $\Gamma = X \cup \bigcup_U \{\text{oscillating curve in U}\}$.
5. For each tooth R_n in the strip S there is a semi-disk S_n that adjoins the tooth from the top (n is even) or the bottom (n is odd) of S.

Key fact

There is $C > 1$ s.t. $S_n \cup R_n$ is a C-quisiconformal image of S_n for all n, and this map equals identity on the circular arc in ∂S_n.

6. Standard machinery: solve the Beltrami equation to make the maps conformal.

Definition

A homeomorphism $h\colon \mathbb{T} \to \mathbb{T}$ is M-**quasisymmetric** if for all adjacent arcs $I, J \subset \mathbb{T}$ of equal length $|\log|h(I)| - \log|h(J)|| \le M$.

Beurling–Ahlfors (1956): h is quasisymmetric $\iff$ h extends to a K-quasiconformal map $\mathbb{D}$ onto itself.

Theorem (Section 4 — new geometric proof)

Every quasisymmetric map is a conformal welding for a K-**quasicircle**, i.e. the image of $\mathbb{T}$ by a K-quasiconformal map $\mathbb{S}^2 \to \mathbb{S}^2$.

Quasicircles

A Γ is a quasicircle iff it satisfies Ahlfors' bounded-turning condition: $\exists C \ge 1$ s.t. for any $z, w \in \Gamma$ and any x on a shorter arc of Γ between z and w $|z - x| + |w - x| \le C\,|z - w|$. Examples: Koch snowflake, Julia sets of $z^2 + c$ near 0.

Proof outline

Bishop's proof is based on reflecting circle chains:

1. Build a symmetric circle chain D_n on n equidistanced points. Let circles of D_n be equal and tangent along $\mathbb{T}$.
2. Using Koebe's theorem construct a circle chain C_n corresponding h.
3. Reflect circle chains repeatedly through each circle. For D_n the limit is $\mathbb{T}$, while C_n limits to a Jordan curve Γ_n.
4. Take the Koebe chain maps f_n for the inner complementary domain to C_n and g_n for the outer one.
5. Pullback f_n and g_n to the inner and outer complementary domains to D_n using universal covering map $\Pi\colon \mathbb{D} \to \mathbb{S}^2 \setminus \{n \text{ points }\}$.
6. Extend $f_n \circ \Pi$ and $g_n \circ \Pi$ to reflected chains by the Schwarz principle, obtaining maps F_n and G_n.
7. The quasiconformal extension H of h conjugates D_n to C_n, giving a K-quasiconformal map of $\mathbb{S}^2$, and has a lift H_n to the universal cover.
8. As $n \to \infty$, C_n converges to a K-quasicircle, and compactness of K-quasiconformal maps allows us to find converging subsequences of $G_n \circ H_n$ and F_n.

Key Takeaways

- **Almost surjectivity.** Every orientation preserving homeomorphism $h\colon \mathbb{T} \to \mathbb{T}$ (if it is not log-singular) agrees with a conformal welding outside a set of arbitrarily small Lebesgue measure *(Theorem 1)*.
- **Capacity is the right language.** The exceptional set where welding fails always has logarithmic capacity zero — this is much stronger than measure zero *(Theorem 2)*.
- **Koebe's theorem is the engine.** The circle-domain theorem provides explicit, geometric approximations through circle chains, replacing abstract existence arguments.
- **Application to quasisymmetric maps.** Circle chains and the reflection trick give a purely geometric proof of the "fundamental theorem of conformal welding" that avoids the Beltrami equation.

Open Problems

Conjecture 1

Every $h\colon \mathbb{T} \to \mathbb{T}$ is a generalized conformal welding on $\mathbb{T} \setminus E$ where E is merely **countable** (strengthening Theorem 2).

Conjecture 2 — Generalized Koebe conjecture

Every Moore decomposition of $\mathbb{R}^2$ is conformally equivalent to a **Koebe decomposition** (all non-trivial elements are disks). This contains the classical Kreisnormierungsproblem as a special case.

Connection between the conjectures

Conjecture 2 $\Rightarrow$ Conjecture 1: apply Conjecture 2 to the decomposition of the annulus A by arcs $\{x \sim h(x)\}$. Since there can only be countably many disjoint disks in the plane, all but countably many arcs collapse to points.

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