

Non-removability of the Sierpiński Gasket



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(Based on Invent paper by Prof. Ntalampekos)

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June 1, 2026

Outline

- 1 Background and main results
- 2 Preliminaries
- 3 Proof of the main theorem
- 4 Open problems

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Quasiconformal removability

Definition (Quasiconformal Removable)

Let $K \subset \mathbb{C}$ be a compact set. K is **quasiconformal removable (QC-removable)** if every homeomorphism $f : \mathbb{C} \rightarrow \mathbb{C}$ that is QC in $\mathbb{C} \setminus K$ is QC in \mathbb{C} .



Quasiconformal Removable in Planar Domain

Definition

We say that a compact set $K \subset U \subset \mathbb{R}^2$ is (quasi)conformally removable inside the domain U if any homeomorphism of U , which is (quasi)conformal on $U \setminus K$, is (quasi)conformal on U .

Basic Fact

A set K is quasiconformally removable inside U if and only if K is conformally removable inside U .

Sobolev removability

Definition (Sobolev Removable)

Let $p \in [1, \infty]$. We say that a compact set $K \subset \mathbb{R}^n$ is $W^{1,p}$ -removable if any real-valued function that is continuous in \mathbb{R}^n and lies in $W^{1,p}(\mathbb{R}^n \setminus K)$, also lies $W^{1,p}(\mathbb{R}^n)$.

Open Problem 1 (Jones-Smirnov, 2000, Ark. Mat.)

Is $W^{1,2}$ -removability in the plane equivalent to quasiconformal removability?

Applications of the removability problem

Open Problem 2

Can we give a geometry characterization of Removability?

Why this problem is important?

Applications

- ▶ Complex Dynamics (quasiconformal surgery) (Shishukura, Sullivan, . . .)
- ▶ Complex Analysis: Conformal Welding
- ▶ Sobolev Analysis: Connections to problems of density and extendability of Sobolev functions (Koskela, Rajala, Zhang, . . .)
- ▶ Probability Theory: SLE and connection to GFF (Duplantier, Miller, Sheffield, . . .)

The brief history of the removability problem

- Removability Problem:
 - ▶ Earlier results: Besicovitch, 1931; Gehring, 1960.
 - ▶ Removability of compact sets: Jones-Smirnov, 1995, 2000; Kaufman-Wu, 1996, 1998; Koskela-Nieminnen 2005; Ntalampekos 2020.
 - ▶ Removability of Julia sets: Kahn, 1998; Graczyk-Smirnov, 2009.
- Counterexamples and Constructions:
 - ▶ Non-removable sets: Kaufman-Wu, 1931; Gehring, 1960.
 - ▶ Constructions of exceptional functions/homeomorphisms: Jones-Smirnov, 1995, 2000; Kaufman-Wu, 1996, 1998; Koskela-Nieminnen 2005; Ntalampekos 2020.
 - ▶ Removability of Julia sets: Kahn, 1998; Graczyk-Smirnov, 2009.

Examples and Counterexamples

Examples of removable sets:

- (Besicovitch, Väisälä) Sets of σ -finite \mathcal{H}^1 -measure.
- Quasicircles: images of circles under quasiconformal mappings of $\hat{\mathbb{C}}$.
- Boundaries of John domains, of Hölder domains, and of domains satisfying certain quasihyperbolic conditions.

Examples of non-removable sets:

- All sets of positive area.
- Bishop's flexible curves, with Hausdorff dimension 1.
- $C \times [0, 1]$ and some product sets $C \times E$, where C, E are Cantor sets.
- Kaufman's graphs, can be α -Hölder continuous with $\alpha < 1/2$.

Removability of fractal structures

Problem

What about sets of more complicated fractal structure?

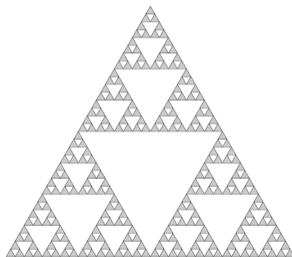


Figure: The Sierpiński gasket.

Question (Bishop): Is the Sierpiński gasket removable?

Main results

The main object of this paper is to prove that the Sierpiński gasket is non-removable for (quasi)conformal maps and Sobolev functions.

The first result answers a question of Bishop:

Theorem 1 (Quasiconformal Non-removability)

The Sierpiński gasket is non-removable for quasiconformal maps.

The second result complements and sharpens the results of Ntalampekos (Math. Z., 2020).

Theorem 2 (Continuous Sobolev Non-removability)

All homeomorphic copies of the Sierpiński gasket are non-removable for continuous Sobolev functions of the class $W^{1,p}$ for $1 \leq p \leq 2$.

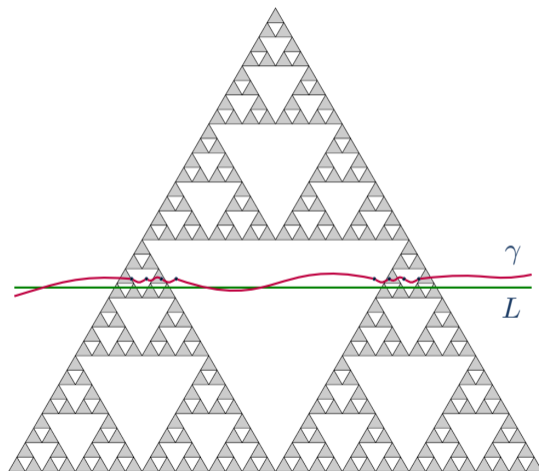
$W^{1,p}$ -removability of the gasket

Theorem (Ntalamepkos, 2020, Math. Z.)

The Sierpiński gasket is $W^{1,p}$ -removable for $p > 2$.

Brief idea of the proof—**Detour property**: For each line L , there exists a detour path γ arbitrarily close to L such that γ intersects only finitely many complementary triangles.

Detour property



Case $p = 2$?

Then there arise a natural question: **What about $p = 2$?**

Theorem (Continuous Sobolev Non-removability)

All homeomorphic copies of the Sierpiński gasket are non-removable for continuous Sobolev functions of the class $W^{1,p}$ for $1 \leq p \leq 2$.

Corollary (Characterization of Sobolev removability of the gasket)

The Sierpiński gasket is $W^{1,p}$ -removable if and only if $p > 2$.

Question (Topological Proof)

Can we give a pure topological proof of the non-removability of the Sierpiński gasket for $1 \leq p \leq 2$?

Quasiconformal non-removability

Theorem (Quasiconformal Non-removability)

The Sierpiński gasket is non-removable for quasiconformal maps, i.e., there exists a homeomorphism $f : \mathbb{C} \rightarrow \mathbb{C}$ that is quasiconformal on $\mathbb{C} \setminus K$ but not quasiconformal on \mathbb{C} .

Question (Extension to all homeomorphic copies)

Is the above theorem works for all homeomorphic copies of the Sierpiński gasket?

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Quasiconformal maps

Definition (Quasiconformality on planar domains)

Let $U, V \subset \mathbb{R}^2$ be open sets. An orientation-preserving homeomorphism $f: U \rightarrow V$ is M -quasiconformal for some $M > 0$ if $f \in W_{\text{loc}}^{1,2}(U)$ and

$$\|Df(z)\|^2 \leq MJ_f(z)$$

for a.e. $z \in U$, where $\|Df(z)\|$ denotes the operator norm of the differential of f at z , and J_f is the Jacobian of f . We also say that f is quasiconformal if it is M -quasiconformal for some $M > 0$. The number $M > 0$ is called the quasiconformal distortion of f .

Quasiconformal maps on metric spaces

Definition (Quasiconformality on metric spaces)

If two metric spaces (X, d_X) and (Y, d_Y) are locally isometric to open subsets of \mathbb{R}^2 , then we say that a homeomorphism $f: X \rightarrow Y$ is M -quasiconformal if the following holds. For each $x \in X$ there exist open neighborhoods U_x of x and $V_{f(x)}$ of $f(x)$ and there exist isometries $\phi: U_x \rightarrow U$ and $\psi: V_{f(x)} \rightarrow V$, where U and V are open subsets of \mathbb{R}^2 such that $\psi \circ f \circ \phi^{-1}: U \rightarrow V$ is M -quasiconformal, in the preceding sense.

Quasisymmetric maps

Definition (Quasisymmetry)

A homeomorphism $f: X \rightarrow Y$ is η -quasisymmetric if there exists a homeomorphism $\eta: [0, \infty) \rightarrow [0, \infty)$ such that for every triple of distinct points $x, y, z \in X$ and for their images $x' = f(x), y' = f(y), z' = f(z)$ we have

$$\frac{d_Y(x', y')}{d_Y(x', z')} \leq \eta \left(\frac{d_X(x, y)}{d_X(x, z)} \right).$$

The function η is called the distortion function associated to f .

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Recall the main theorem

Theorem (Quasiconformal Non-removability)

The Sierpiński gasket is non-removable for quasiconformal maps, i.e., there exists a homeomorphism $f : \mathbb{C} \rightarrow \mathbb{C}$ that is quasiconformal on $\mathbb{C} \setminus K$ but not quasiconformal on \mathbb{C} .

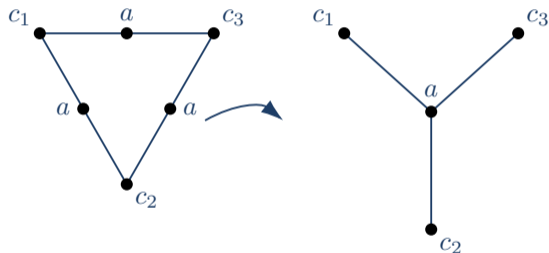
Our main target is to construct such a homeomorphism $f : \mathbb{C} \rightarrow \mathbb{C}$!

Proof of the main theorem

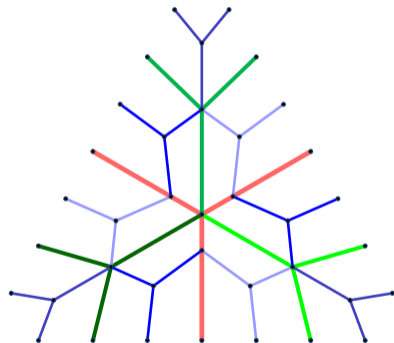
Brief idea of the proof—construction of the **flap-planes**.

- (1) Step 1: Collapse triangles to tripods continuously.
- (2) Step 2: Construct an abstract surface.
- (3) Step 3: The exceptional homeomorphism.

Step 1: Collapse triangles to tripods continuously



Step 1: Collapse triangles to tripods continuously



Step 1: Collapse triangles to tripods continuously

We obtain a map $f : \mathbb{C} \rightarrow \mathbb{C}$:

- **continuous** and **surjective**;
- **injective** outside all the triangles;
- $f(K)$ has **full Lebesgue measure**.

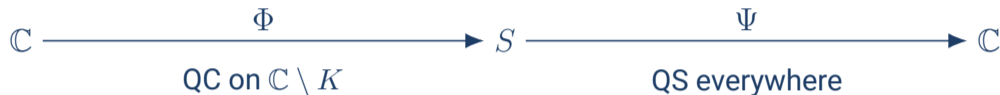
Step 2: Construct an abstract surface

The most technical and intriguing step in this paper:

We construct an abstract surface S and extend the map $f : \mathbb{C} \rightarrow \mathbb{C}$ to a homeomorphism $\Phi : \mathbb{C} \rightarrow S$:

- Φ is quasiconformal outside K ;
- $\Phi(K)$ has **positive** \mathcal{H}^2 measure in S ;
- S is **Ahlfors 2-regular**: $\mathcal{H}^2(B(x, r)) \sim r^2$;
- S is a **quasiplane**: there exists a quasisymmetric map $\Psi : S \rightarrow \mathbb{C}$.

Step 3: The exceptional homeomorphism



- $\Psi \circ \Phi$ is a homeomorphism, quasiconformal on $\mathbb{C} \setminus K$
- $\Psi \circ \Phi$ is not quasiconformal on \mathbb{C} :

$$K \mapsto \Phi(K) \quad (\text{positive } \mathcal{H}^2\text{-measure})$$

$$\Phi(K) \mapsto \Psi \circ \Phi(K) \quad (\text{positive area})$$

Quasisymmetric embedding into the plane: The map Ψ

Theorem (Bonk-Kleiner, 2002, Invent. Math.)

Let (X, d) be an Ahlfors 2-regular metric space homeomorphic to \mathbb{S}^2 . Then (X, d) is quasisymmetric to \mathbb{S}^2 if and only if (X, d) is linearly locally connected.

We recall the following settings in the theorem:

- (1) **Ahlfors 2-regular:** $\mathcal{H}^2(B(x, r)) \sim r^2$;
- (2) **Linearly locally connected (LLC):** If there is a constant $C \geq 1$ so that every pair of points in $B(x, R)$ can be joined in $B(x, CR)$ and every pair of points in $X \setminus B(x, R)$ can be joined in $X \setminus B(x, R/C)$, whenever $B(x, R)$ is a ball in X . (no cusps and no tall waves/wrinkles).

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Open problems

Problem 1

If a set is quasiconformally removable, is it also locally quasiconformally removable?

Problem 2

Is the union of two intersecting compact sets quasiconformally removable, whenever each one of them is removable?

Problem 3

Characterize the quasiconformal removable sets in dimension 2. Is this equivalent to *CNE* sets defined by Ntalampekos?

Thank you!