# Transboundary Extremal Length (based on O. Schramm)

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# For those who read in advance

(*) – some reminders that I will skip if we'll see they in previous talks
(**) – proofs that I will give on the blackboard only if time permits
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# Conformal Equivalence (*)

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## Definition

Let $U$ and $V$ be open subsets of $\mathbb{C}$. We say that they are conformally equivalent if there exists a biholomorphism $f : U \to V$.

# Very Important Example (*)

**Linear fractional transformation (Möbius transformation):**

$$z \mapsto \frac{az+b}{cz+d},$$

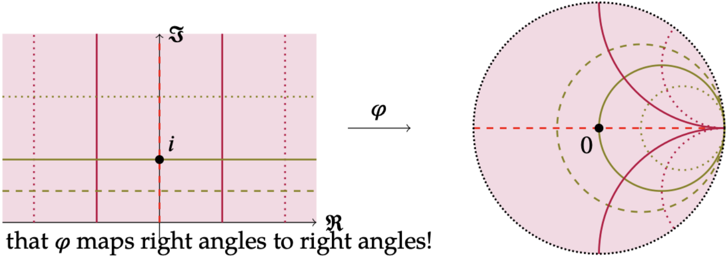
where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

# (*)

The **Cayley transform**

$$z \mapsto \frac{z-i}{z+i}$$

gives a biholomorphism between the upper half-plane $\mathbb{H} = \{ z : \Im(z) > 0 \}$ and the unit disk $\mathbb{D} = \{ z : |z| < 1 \}$.




Note that $\varphi$ maps right angles to right angles!

# Automorphismes (*)

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**Proposition**

*Automorphismes of $X$ are of the form $Y$.*

| $X$ | $\mathbb{C}$ | $\widehat{\mathbb{C}}$ | $\mathbb{D}$ | $\mathbb{H}$ |
|---|---|---|---|---|
| $Y$ | $f(z) = az + b$<br>$a, b \in \mathbb{C}$<br>$a \neq 0$ | $f(z) = \frac{az+b}{cz+d}$<br>$a, b, c, d \in \mathbb{C}$<br>$ad - bc \neq 0$ | $f(z) = \frac{az+b}{\bar{b}z+\bar{a}}$<br>$a, b \in \mathbb{C}$<br>$\lvert a\rvert^2 - \lvert b\rvert^2 = 1$ | $f(z) = \frac{az+b}{cz+d}$<br>$a, b, c, d \in \mathbb{R}$<br>$ad - bc = 1$ |

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# Conformal Uniformization Theorems

Conformal mappings preserve important geometric properties, however, usually it's easier to prove properties for "nice" domains of $\mathbb{C}$ (or $\widehat{\mathbb{C}}$). It leads to following question.

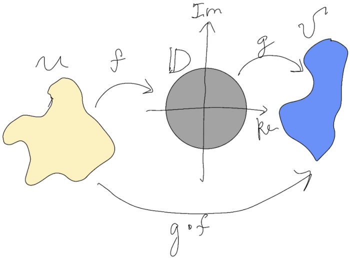
**Question**

*Which domains of the complex plane $\mathbb{C}$ (or of the Riemann sphere $\widehat{\mathbb{C}}$) are conformally equivalent?*

# Riemann Mapping Theorem

**Theorem (Riemann Mapping Theorem. B. Riemann (1851), W. Osgood (1900).)**

*Let $U \subsetneq \mathbb{C}$ be a simply connected open subset which is not $\mathbb{C}$. Then $U$ is conformally equivalent to the open unit disk $\mathbb{D} = \{ z : |z| < 1 \}$.*




**Corollary**

*If $U, V \subsetneq \mathbb{C}$ are open simply connected subsets of $\mathbb{C}$, then they are conformally equivalent.*

# Koebe's conjecture

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General case – still open

# (Transboundary) Extremal Length

$D$ – domain in $\mathbb{C}$, $\Gamma$ – collection of rectifiables curves in $D$.
For a non-negative Borel-measurable function $\rho : D \longrightarrow \mathbb{R}_+$ and a rectifiable curve $\gamma$ the *the $\rho$-length* of $\gamma$ is

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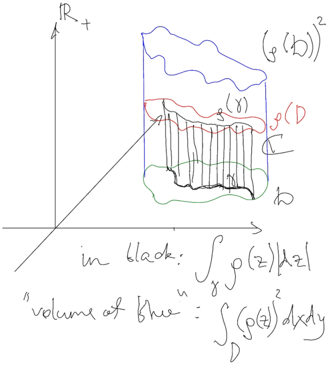
and the *area* of $\rho$ is

$$A(\rho) := \int_D \rho^2 dLeb.$$

# Extremal Length

The extremal length of Γ is defined as

$$EL_D(\Gamma) := \sup_{\rho} \frac{L_\rho(\Gamma)^2}{A(\rho)} = \sup_{\rho} \frac{(\inf_{\gamma\in\Gamma} \int_\gamma \rho|dz|)^2}{\int_D \rho^2 dLeb}$$



# Conformal Invariance of Extremal Length

Extremal length is conformly invariant. Namely, let $f : D \longrightarrow D^*$ be a conformal homeomorphisme. Then for every $\Gamma = \{ \gamma \text{ curves in } D \}$ one has $EL(\Gamma) = EL(\Gamma^*)$, where $\Gamma^* = \{ f \circ \gamma : \gamma \in \Gamma \}$.

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The conformal invariance of extremal length provides the following.

**Theorem**

*Annulus $A_1$ and $A_2$ of inner radius $r_i$ and outer radius $R_i$ are conformly equivalent if and only if $\frac{R_1}{r_1} = \frac{R_2}{r_2}$.*

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**Proof.**

(<=) The biholomorphisme is given by a dilation $z \mapsto \frac{R_2}{R_1} z$.
(=>) Hint: for an annulus $A$ with radius $(r, R)$ and $\Gamma$ the set of all (regular) curves from outer circle to inner circle $EL_A(\Gamma) = \frac{R}{r}$ (**). □

# Quasiconformal mappings

There are different equivalent definitions of quasiconformal mappings, we follow this one.

## Definition

The mapping $f : D \longrightarrow D^*$ between two domains in $\mathbb{C}$ is said to be $K$-quasiconformal if for every collection $\Gamma$ of curves in $D$ one has

$$\frac{EL_D(\Gamma)}{EL_{D^*}(\Gamma^*)} \leq K,$$

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Informally – takes infinisemal circles to infinisemal ellipses of eccentricity $\leq K$; equivalently, permit the bounded distortion of angles locally.

# Example

some mor examples and explanations from quasiconformal surgery example of smth quasi conformal x,y -¿ x, 2y

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How: the ends compactification of a domain $D$, “shrinking” connected components of the complement to points

# The Ends Compactification

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## Definition

Let $D$ be any domain in $\widehat{\mathbb{C}}$. Then its *ends compactification*

$$\mathcal{E}(D) = \widehat{\mathbb{C}} / \underset{x,y \in \text{the same c.c. of } \widehat{\mathbb{C}} \setminus D}{\overset{\sim}{x \sim y \text{ iff}}}$$

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with the quotient topology.

Notations: $\pi_D : \widehat{\mathbb{C}} \longrightarrow \mathcal{E}(D)$ – the quotient map; $[p] = \pi_D^{-1}(p)$;
$\mathcal{C}(D) = \mathcal{E}(D) \setminus D$ – the space of complementary components;
$p \in \mathcal{C}(D)$ insubstantial (substantantial) compl. component if $[p]$ contains (more than) one point.

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**Exercise**

The transboundary extremal length of $\Gamma$ is equal to the classical extremal length when all curves of $\Gamma$ lie in $D$.

# Fat and Cofat Sets

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## Definition

A set $A \subset \widehat{\mathbb{C}}$ is called *$\tau$-fat* if for every $x \in A \cap \mathbb{C}$ and for every disk $B = \mathbb{B}(x, r)$ that does not contain $A$ we have $\text{area}(A \cap B) \geq \tau \, \text{area}(B)$.
The set $A$ is *fat* if it is $\tau$-fat for some $\tau > 0$.
A domain $D \subset \widehat{\mathbb{C}}$ is $\tau$-cofat if each connected component of its complement is $\tau$-fat. $D$ is *cofat* if it is $\tau$-cofat for some $\tau > 0$

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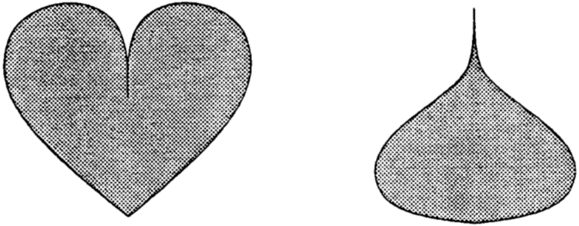

Figure 2.1.a. A heart is fat, b. an outward cusp is not.

# More fat sets

Disks are (1/4)-fat. (**)
Moreover, they are the only compact subsets of $\mathbb{C}$ that contain more than a single point and are (1/4)-fat. (**)

# Fatness Invariance

Fatness is Mobius invariant and quasiconformally invariant.
Quasidisks are fat.

# Kernel Convergence

Will i talk about KCK and extended KCK or i'll pass to uniformization thm directly without any proof?

# Extended Kernel Convergence

**Theorem**

*Let $D \subset \widehat{\mathbb{C}}$ be a $\tau$-cofat domain, $\tau > 0$, and let $f_n : D \longrightarrow \widehat{\mathbb{C}}$ be a sequence of conformal maps that converges to a conformal map $f$. Suppose that for each $n$ there is a domain $D_n$ containing $D$ such that*

1. *$D_n$ is a union of $D$ and a collection of connected components of $\widehat{\mathbb{C}} \setminus D$.*
2. *$\mathcal{C}(D_n)$ is at most countable.*
3. *$f_n$ extends to a conformal mapping $\widehat{f}_n : D_n \longrightarrow \widehat{\mathbb{C}}$ and $\widehat{f}_n(D_n)$ is $\tau$-cofat.*

*Let $b \in \mathcal{C}(D)$.*

# Uniformisation of cofat domains

## Theorem

*Let* $D \subset \widehat{\mathbb{C}}$ *be a cofat domain. ..*
*Then ...*
*Furtheremore*

# Uniformisation in the countable case