

# Every Circle Homeomorphism is the Composition of Two Weldings

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## Definitions

### Definition

Let  $\gamma \in \hat{\mathbb{C}}$  be a Jordan curve in the Riemann sphere, and let  $\Omega, \Omega'$  be the two complementary components of  $\gamma$ . A **Conformal Welding** is a map  $h = g^{-1} \circ f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ , where  $f : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow \Omega$  and  $g : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow \Omega'$  are conformal maps.

### Definition

For the sake of brevity, we will refer to orientation-preserving homeomorphisms  $\phi : S^1 \rightarrow S^1$  (where  $S^1$  is the unit circle) as **Orientation-Preserving Circle Homeomorphisms** or simply **Circle Homeomorphisms**.

## Figural depiction

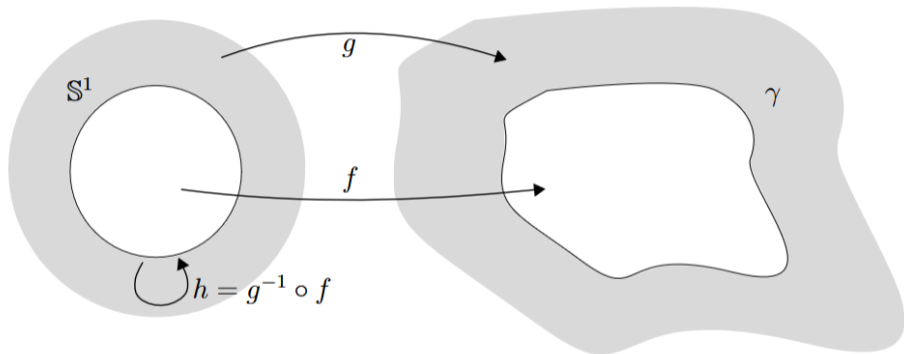


Figure: Depiction of a conformal welding  $h$  via the curve  $\gamma$

(Image credit: Alex Rodriguez)

# Main Result

## Theorem (Rodriguez, 2026)

*Every orientation-preserving circle homeomorphism is the composition of two conformal weldings.*

## Are Circle Homeomorphisms Weldings?

- Not always, because weldings are not preserved under composition (Vaimo, 1985)
- Not all circle homeomorphisms are weldings. Example:

$$\phi(x) = \begin{cases} x & \text{for } x \leq 0 \\ x^3 & \text{for } x \geq 0 \end{cases}$$

So, the main theorem of this paper provides the above result as a corollary.

- However, if a circle homeomorphism is quasimetric, then it is always a welding.

### Theorem (Pfluger, 1960)

*Every quasi-symmetric homeomorphism  $\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is a conformal welding.*

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### **Theorem (Pfluger,1960)**

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Let  $\mu$  be a finite, compactly supported signed Borel measure. If

$$I(\mu) := \int \int \log \frac{1}{|z - \xi|} |d\mu|(\xi) |d\mu|(z) < \infty$$

Then we say that  $\mu$  has *finite energy* and refer to  $I(\mu)$  as the *energy integral* of  $\mu$ . Let  $K \subset \mathbb{C}$  be compact and let  $P(K)$  be the set of all Borel probability measures on  $K$ . Define:

$$\gamma(K) = \inf\{I(\mu) : \mu \in P(K)\}$$

Then, we define the **Logarithmic Capacity** and the **Capacity** of  $K$  to be

$$\widetilde{Cap}(K) = e^{-\gamma(K)} \quad Cap(K) = 1/(\gamma(K) + \log(2)),$$

respectively. We define the capacity of a Borel set  $E$  to be:

$$Cap(E) = \sup\{Cap(K) : K \text{ compact}, K \subset E\}.$$

## More on Logarithmic Capacity

- Note that  $Cap(K) = 0$  IFF  $\widetilde{Cap}(K) = 0$ .
- Examples:
  - ▶ For a disk  $D(z, r)$ ,  $\widetilde{Cap}(D(z, r)) = r$ ,  $Cap(D(z, r)) = 1/\log(2/r)$ .
  - ▶ For a line segment  $L \subset \mathbb{C}$  of length  $h$ ,  $\widetilde{Cap}(L) = h/4$ ,  $Cap(L) = 1/\log(8/h)$
- 'Nice' maps (such as bi-Holder maps) preserve sets of zero capacity.

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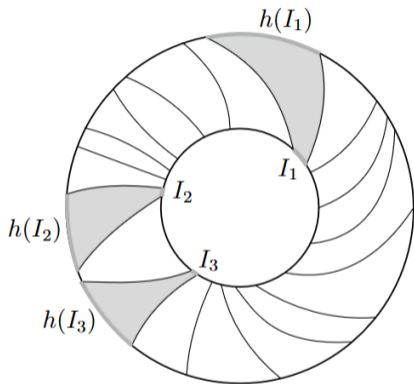
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- 'Nice' maps (such as bi-Holder maps) preserve sets of zero capacity.

**Definition**

A homeomorphism  $h : A \rightarrow B$ ,  $A, B \subset \mathbb{C}$  is **log-singular** if there exists a Borel set  $E \subset A$  such that  $Cap(E) = 0$  and  $Cap(h(A \setminus E)) = 0$ .



**Figure:** A representation of a log-singular homeomorphism  $h$ , which maps the arcs  $I_j$  of small capacity to arcs  $h(I_j)$  of large capacity. (Image credit: Alex Rodriguez)

## Relation to Main Theorem

### Theorem (Bishop, 2007)

*Every log-singular circle homeomorphism  $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is a conformal welding.*

- So it is enough to prove:

### Theorem (Rodriguez)

*Every circle homeomorphism is the composition of two log-singular maps.*

- More specifically:

### Theorem (Rodriguez)

*Let  $\phi : S^1 \rightarrow S^1$  be a circle homeomorphism. Then, there exists a log-singular map  $h : S^1 \rightarrow S^1$  such that  $\phi \circ h^{-1}$  is log-singular.*

- Thus,  $\phi = (\phi \circ h^{-1}) \circ h$  is the composition of two conformal weldings.

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## Definition

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A **log-singular set** is a set  $E \subset \mathbb{S}^1$  such that  $Cap(E) = 0$  and such that there exists a log-singular map  $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  with  $Cap(h(\mathbb{S}^1 \setminus E)) = 0$

We can describe log-singular sets and their related log singular maps using addresses. Addresses encode partitions of an interval or arc of  $\mathbb{S}^1$ .

**Example:** Dyadic partitions of  $[0, 1]$ :

$$\left\{ \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right] : k \in \mathbb{N}, 0 \leq k \leq 2^n \right\}$$

We will define addresses on the interval  $I = [0, 1]$ . This definition can be applied to  $\mathbb{S}^1$  or a subarc thereof in a similar way.

## Addresses

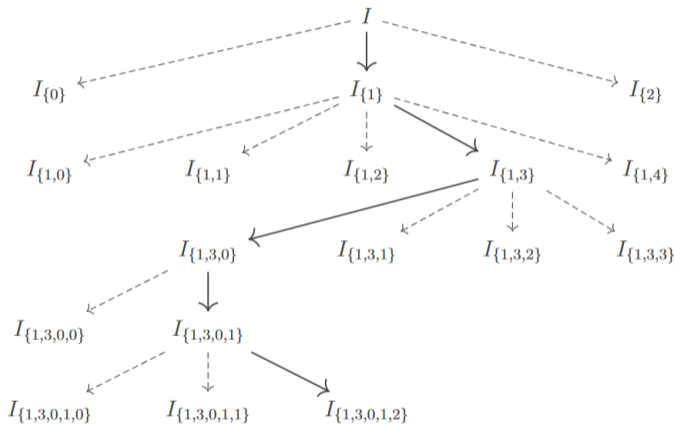


Figure: Addresses associated with a partition of an interval  $I$ ; with  $L_1 = 3$ ,  $L_2 = 5$ ,  $L_3 = 4$ ,  $L_4 = 2$ ,  $L_5 = 3$ . The address to  $I_{\{1,3,0,1,2\}}$  is shown in non-dashed arrows.

**Definition**

Let  $\{L_n\}_{n \geq 0}$ ,  $L_n \in \mathbb{Z}^+ \forall n$ ,  $L_0 = 1$  be a sequence. A *word* of length  $n$  is a set of the form  $A_n = \{a_1, a_2, \dots, a_n\} \in \mathbb{Z}^n$ , where  $0 \leq a_j < L_j$  for  $1 \leq a_j < L_j \forall 1 \leq j < n$ . Let  $\mathcal{A}_n$  be the set of all possible words  $A_n$  of length  $n$ , and let  $\mathcal{A}_\infty$  be the set of all infinite words  $A_\infty$ . Suppose that  $\forall n \geq 1$  we have intervals  $I_{A_n} \subset [0, 1]$  satisfying:

1. If  $n \leq m$  and  $A_n = \{a_1, a_2, \dots, a_n\}$ ,  $\tilde{A}_m = \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n, \dots, \tilde{a}_m\}$ , then  $I_{\tilde{A}_m} \subset I_{A_n}$  IFF  $a_j = \tilde{a}_j \forall 1 \leq j \leq n$ .
2. If  $A_n \neq \tilde{A}_n$ , then  $I_{A_n}$  and  $I_{\tilde{A}_n}$  have disjoint interiors.
3.  $L_n$  is equal to the number of intervals into which each  $I_{A_{n-1}}$  has been divided, which is the same for all  $I_{A_{n-1}}$ .
4. For each  $n$ ,  $\cup_{A_n \in \mathcal{A}_n} I_{A_n} = I$ . Moreover, for any given  $I_{A_{n-1}}$ :

$$I_{A_{n-1}} = \bigcup_{0 \leq a < L_n} I_{\{a_1, a_2, \dots, a_{n-1}, a\}}$$

Then, we say that  $A_n$  is the **Address** of  $I_{A_n}$ . We also define  $A_n(a) = \{a_1, a_2, \dots, a_n, a\}$ .

## Lemma 3.2

### Lemma (Rodriguez, 2026)

Let  $I, J \subset \mathbb{S}^1$  be subarcs, and let  $\{L_n\}_{n \in \mathbb{N}}$  be a sequence such that  $L_0 = 1$  and  $L_n$  is a positive, even integer for all  $n \geq 1$ . Consider words  $A_n = \{a_1, a_2, \dots, a_n\}$  with  $0 \leq a_j < L_j$ . Suppose that  $\{I_{A_n}\}$  is a partition of  $I = I_{A_0}$ , and  $\{J_{A_n}\}$  is a partition of  $J = J_{A_0}$  such that  $\forall \epsilon > 0, \exists N \geq 1$  such that  $\forall n \geq N$  and  $\forall A_n, |I_{A_n}| + |J_{A_n}| < \epsilon$ . Define:

$$E = \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \left( \bigcup_{A_n \in \mathcal{A}_n} \bigcup_{j=1}^{L_{n+1}/2} I_{A_n(2j-1)} \right), \quad F = \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} \left( \bigcup_{A_n \in \mathcal{A}_n} \bigcup_{j=1}^{L_{n+1}/2} J_{A_n(2j-2)} \right)$$

If  $\text{Cap}(E) = 0$  and  $\text{Cap}(F) = 0$ , then there exists a log-singular map  $h : I \rightarrow J$  such that  $h(I \setminus E) = F$ .

## Lemma 4.1

### Lemma

Let  $\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  be an orientation-preserving circle homeomorphism. Then, there exists a partition  $\{I_{A_n}\}$  of  $\mathbb{S}^1$  and a log-singular set:

$$E = \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \left( \bigcup_{A_n \in \mathcal{A}_n} \bigcup_{j=1}^{L_{n+1}/2} I_{A_n(2j-1)} \right)$$

such that  $\text{Cap}(\phi(E)) = 0$ . Here, the sequence  $\{L_n\}$  associated with  $\{A_n\}$  satisfies  $L_0 = 1$  and  $L_n$  is even for all  $n \geq 1$ .

## Sketch of Proof of Main Theorem

- Let  $\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  be a circle homeomorphism.
- By the second Lemma (Lemma 4.1), there exists a log-singular set  $E \subset \mathbb{S}^1$  such that  $Cap(\phi(E)) = 0$ .
- Because  $E$  is log-singular, there exists a log-singular homeomorphism  $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  such that  $Cap(h(\mathbb{S}^1 \setminus E)) = 0$ .
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Thus,  $\phi \circ h^{-1}$  is log-singular.

- Since log-singular maps are conformal weldings,  $\phi = (\phi \circ h^{-1}) \circ h$  is the composition of two conformal weldings.

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# Open Problems

- is the composition of two bi-holder weldings a welding?
- is the set of weldings Borel?

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