Geometric Function Theory Conference

Malik Younsi: Removability, rigidity of cicle domains and Koebe's Conjecture

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Overview

What this talk is about

1. **The Big Picture**
   Why circle domains? What is Koebe's problem?
2. **Measurable Reimann Mapping Theorem and Removability**
   What does it mean and how we can use it as a tool?
3. **Rigidity**
   What does it mean to be "rigid"? When does it fail?
4. **Main Results**
   Equivalence of two conjectures; conformal $\Leftrightarrow$ QC rigidity
5. **Key New Technique**
   Trans-quasiconformal deformation of Schottky groups

The Starting Point: Mapping Domains

Conformal equivalence — preserving angles, not distances

A fundamental question in complex analysis

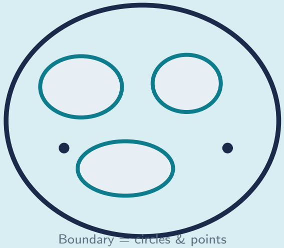
- Can we "simplify" a complicated domain by mapping it *conformally* onto a nicer one?
- **Riemann's theorem:** any simply-connected domain maps to the unit disk.
- But what about domains with *holes*?

Circle Domain

A domain $\Omega \subset \widehat{\mathbb{C}}$ is a **circle domain** if every boundary component is either a *circle* or a *point*.

Circle domains are the natural "normal form" for domains with holes.

Circle Domain Ω



Koebe's Uniformization Problem

A 100+ year old open problem

Theorem (Koebe, 1918 — finitely-connected case)

Any finitely-connected domain in $\widehat{\mathbb{C}}$ is conformally equivalent to a circle domain, uniquely up to Möbius transformations.

Theorem (He–Schramm, 1993)

Any domain with *at most countably many* boundary components is conformally equivalent to a circle domain, uniquely up to Möbius transformations.

Koebe's Conjecture (1909 — still open!)

Every domain in $\widehat{\mathbb{C}}$ is conformally equivalent to a circle domain.

Key insight from He–Schramm: **uniqueness** *(rigidity) is an essential ingredient in the proof of existence.*

Preliminaries and conformal extensions

Quasiconformal map

Let $K \geq 1$, let U, V be domains in $\hat{\mathbb{C}}$ and let $f : U \to V$ be an orientation-preserving homeomorphism. We say that f is K-quasiconformal if it belongs to the Sobolev space $W^{1,2}_{loc}(U)$ and satisfies the Beltarmi equation

$$\partial_{\bar{z}} f = \mu \partial_z f \qquad (1)$$

almost everywhere on U, for some measurable function $\mu : U \to \mathbb{D}$ with $\|\mu\|_\infty \leq \frac{K-1}{K+1}$.

A mapping is **conformal** if and only if it is 1-quasiconformal (Weyl's Lemma).

Some properties

- Set of quasiconformal mappings has a multiplicative group structure.
- Satisfies Lusin properties.

Measurable Reimann mapping theorem and removability

MRMT

Let U be a domain in $\hat{\mathbb{C}}$ and let $\mu : U \to \mathbb{D}$ be a measurable function with $\|\mu\|_\infty \leq 1$. Then there exists a quasiconformal mapping f on U such that $\mu = \mu_f$, i.e

$$\partial_{\bar{z}} f = \mu \partial_z f \tag{2}$$

almost everywhere on U. Moreover, the map f is unique up to post-composition by a conformal map, in the sense that a quasiconformal mapping g on U satisfies $\mu_g = \mu = \mu_f$ if and only if $fog^{-1} : g(U) \to f(U)$ is conformal.

Removability

A compact set $E \subset \mathbb{C}$ is **conformally removable (quasiconformally removable)** if every homeomorphism of $\widehat{\mathbb{C}}$ that is conformal (quasiconformal) outside E is actually a conformal (quasiconformal) everywhere.

Removable

- Quasicircles
- Sets of σ-finite length
- Many sets of Hausdorff dim. < 2

Not removable

- Positive-area sets
- Some Cantor sets of dim. 1
- Ahlfors–Beurling examples (1950)

A remarkable consequence of the MRMT is that the notion of quasiconformal removability and conformal removability actually coincide.

Proposition 9

A compact set $E \subset \mathbb{C}$ is quasiconformally removable if and only if it is conformally removable.

Local property of removability, Proposition 11

Let $E \subset \mathbb{C}$ be compact. Then the following are equivalent:

- For any open set U with $E \subset U$, every homeomorphism $f : U \to f(U)$ which is conformal (quasiconformal) on $U \setminus E$ is actually conformal (quasiconformal) on the whole open set U;
- E is conformally (quasiconformally) removable.

Conformal Rigidity

Conformal Rigidity

A circle domain Ω is **conformally rigid** if every conformal map of Ω onto another circle domain is the restriction of a *Möbius transformation*.

✓ Rigid

- Any finitely-connected circle domain *(Koebe)*
- Countably many boundary components *(He–Schramm '93)*
- σ-finite length boundary *(He–Schramm '94)*

× Not always rigid

Complements of *certain Cantor sets* in the boundary can admit non-Möbius conformal maps.

Rigidity breaks down when the boundary is "too wild."

Understanding rigidity is believed to provide substantial insight into Koebe's conjecture.

The Rigidity Conjecture (He–Schramm)

Connecting rigidity and removability

Rigidity Conjecture (He–Schramm, 1994)

For a circle domain Ω, the following are **equivalent**:

(A) Ω is conformally rigid;

(B) The boundary $\partial\Omega$ is conformally removable;

(C) Every Cantor set contained in $\partial\Omega$ is conformally removable.

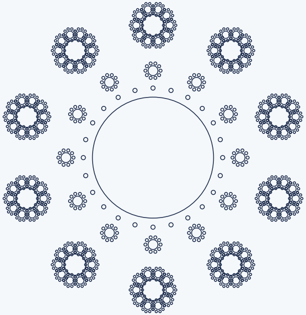
The present paper is devoted to the study of this conjecture. The main result is the following

Theorem 3

Let Ω be a circle domain whose boundary is a *countable union* of circles, Cantor sets, and isolated points. Then (B) and (C) are equivalent.

An example

Domain that does not satisfies the assumptions



Theorem 3 is a direct consequence of the following result on unions of certain removable sets, which is of independent interest. Also, this is the first non-trivial result on union of removable sets.

Theorem 4

Let E be a compact plane set of the form

$$E = \bigcup_{j=1}^{\infty} \Gamma_j \cup \bigcup_{k=1}^{\infty} C_k \cup \bigcap_{l=1}^{\infty} \{z_l\} \tag{3}$$

where each Γ_j is a quasicircle, each C_k is a cantor set and each z_l is a complex number. Then E is conformally removable if and only if every C_k is conformally removable.

The next result shows the equivalence in rigidity.

Theorem 5

A circle domain Ω is **conformally rigid** if and only if it is **quasiconformally rigid**.

Conformal rigidity

Every conformal map $\Omega \to \Omega'$ (circle domain) is the restriction of a Möbius transformation.

$\Leftrightarrow$

Quasiconformal rigidity

Every QC map $\Omega \to \Omega'$ extends to a QC homeomorphism of the whole sphere.

Corollary 6

If Ω is conformally rigid and $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is quasiconformal with $f(\Omega)$ a circle domain, then $f(\Omega)$ is also conformally rigid.

Proof of Theorem 4

- If E is conformally removable, then clearly every C_k, as a compact subset of E, is also conformally removable.
- Assume every C_k is conformally removable. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a homeomorphism which is conformal on $D := \widehat{\mathbb{C}} \setminus E$. Let D' be the union of all open subset of $\widehat{\mathbb{C}}$ on which f is conformal. We show that $D' = \widehat{\mathbb{C}}$, i.e, $E' := \widehat{\mathbb{C}} \setminus D' = \emptyset$.
- Assume that E' is not empty.
- By Baire category theorem there is some $j \in \mathbb{N}$ such that $\Gamma_j \cap E$ has empty interior in E'.
- Then there exists open set $U \subset \mathbb{C}$ such that $U \cap E' \neq \emptyset$ and $U \cap E' \subset \Gamma_j \cap E'$.
- By using locality, we show that f is conformal on U. Hence f is conformal on $U \cup D'$, which contradict the maximality.

Techniques for Theorem 5: Trans-QC Deformation of Schottky Groups

Schottky Group $\Gamma(\Omega)$

For a circle domain with boundary circles $\gamma_1, \gamma_2, \ldots$, the **Schottky group** is generated by the reflections R_j across each γ_j. More precisely, for $j \in \mathbb{N}$, $R_j : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ the refection across the circle γ_j:

$$R_j(z) = a_j + \frac{r_j^2}{z - a_j} \tag{4}$$

where a_j is the center and r_j is the radius of the circle γ_j.
$\Gamma(\Omega)$ consists of the identity map and compositions of R_j.

David coefficient

We say that a measurable function $\mu : U \to \mathbb{D}$ is a David coefficient if there exist constant $M > 0$, $\alpha > 0$ and $0 < \epsilon_0 < 1$ such that

$$m(\{z \in U : |\mu(z)| > 1 - \epsilon\}) < M \exp -\frac{\alpha}{\epsilon} \tag{5}$$

where m is two dimensional Lebesgue measure.

David map

An orientation-preserving homeomorphism f on U is called David map if

- $f \in W^{1,1}_{loc}(U)$
- satisfies Beltarmi equation almost everywhere on U
- for some measure function $\mu : U \to \mathbb{D}$ satisfies (5)

In this case the function μ is called the David coefficient of f and is denoted by μ_f.

David integrability theorem

Let U be a domain in $\widehat{\mathbb{C}}$ and let $\mu : U \to \mathbb{D}$ be a David coefficient o U. Then there exists a David map f on U such that $\mu = \mu_f$, i.e,

$$\partial_{\overline{z}} f = \mu \partial_z f$$

almost everywhere on U. Moreover, the map f is unique up to post-composition by a conformal map, in the sense that a David map g on U satisfies $\mu_g = \mu = \mu_f$ if and only if $f \circ g^{-1} : g(U) \to f(U)$ is conformal.

Pullback

Let $V \subset \mathbb{C}$ be open and let $\mu : V \to \mathbb{D}$ be measurable. If $f : U \to V$ is an orientation-preserving strongly David map, then one can define a measurable function $f^\star(\mu : U \to \mathbb{D})$, called pullback of μ by f, by

$$f^\star(\mu) := \frac{\partial_{\overline{z}} f + (\mu \circ f)\overline{\partial_z f}}{\partial_z f + (\mu \circ f)\overline{\partial_{\overline{z}} f}}.$$

Invariant with respect to $\Gamma(\Omega)$

We say that a measurable function $\mu : \widehat{\mathbb{C}} \to \mathbb{D}$ is invarient with respect to $\Gamma(\Omega)$ if $T^\star(\mu) = \mu$ almost everywhere $T \in \Gamma(\Omega)$.

Proposition 16

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a strongly David map whose strongly David coefficient μ_f is invariant with respect to the Schottky group $\Gamma(\Omega)$. Then $f(\Omega)$ is a circle domain whose corresponding Schottky group is $f\Gamma(\Omega)f^{-1}$.

Coming back to the proof of Theorem 5

Before we proceed with the proof of Theorem 5, we need the following two lemmas.

Lemma 18

Let Ω be a circle domain in $\widehat{\mathbb{C}}$. If Ω is quasiconformally rigid, then $\partial\Omega$ has zero area.

Lemma 19(He and Schramm)

Let Ω be a circle domain, let P be the set of its point boundary components and let $\Omega' = \Omega \cup P$. Suppose that $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasiconformal mapping which maps Ω onto a circle domain $g(\Omega)$. Let μ be the Beltarmi coefficient of g restricted to Ω', and denote by $\tilde{\mu}$ its invariant extension to $\widehat{\mathbb{C}}$. Suppose that $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasiconformal mapping with $\mu_h = \tilde{\mu}$ almost everywhere. Then $g \equiv T \circ h$ on Ω for some Möbius transformation T.

Proof of Theorem 5

Let Ω be a circle domain. We want to show that Ω is conformally rigid if and only if it is quasiconformally rigid.

Steps ($\Longrightarrow$):

- Assume that Ω is conformally rigid., and let $f : \Omega \to f(\Omega)$ be quasiconformal, wher $f(\Omega)$ is a circle domain.
- Set $\mu := \mu_{f^{-1}}$ on Ω and consider the invariant extension $\tilde{\mu}$ of μ.
- BY MRT, there is a quasiconformal mapping $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\mu_g = \tilde{\mu}$ almost everywhere on $\widehat{\mathbb{C}}$.
- Then $g \circ g$ is conformal on Ω and by using Proposition 16, $(g(f(\Omega))$ is circle domain.
- Since Ω is conformally rigid, we get that $g \circ f$ is the restriction of a Möbius transformation. This conclude that Ω is quasiconformally rigid.

Steps for converse:

- Assume that Ω is quasiconformally rigid, and let $f : \Omega \to f(\Omega)$ be conformal, where $f(\Omega)$ is a circle domain.
- In particular f is quasiconformal on Ω, hence is the restriction of a quasiconformal mapping g of the whole sphere, by quasiconformal rigidity of Ω.
- By Lemma 18, the $\partial\Omega$ has zero area and thus the Beltarmi coefficient of g restricted to Ω' is zero almost everywhere.
- From Lemma 19, the map $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a Möbius transformation.
- Since $f = g = T \circ h$ on Ω, we get that f is the restriction of a Möbius transformation, which implies that Ω is conformally rigid.

Summary & Open Questions

What was proved, and what remains

Results in this paper

- (B) $\Leftrightarrow$ (C) for large class of circle domains *(Thm 3, 4)*
- Conformal rigidity $\Leftrightarrow$ QC rigidity *(Thm 5)*
- Rigid domains are QC-invariant *(Cor. 6)*
- First result on unions of removable sets

Thank You!

Open questions

- Does (A) $\Leftrightarrow$ (B) hold in general? *(Full Rigidity Conjecture)*
- Is the union of two removable sets removable?
- Do there exist Cantor sets that are conformally removable but whose complement is not rigid, fundamentally different from Ahlfors–Beurling?
- Can Koebe's conjecture be proved for uncountably many boundary components?