Conformal weldings of flexible curves

Kristina Oganesyan

Steklov Mathematical Institute

Student Conference on Geometric Function Theory
July 2026

Conformal weldings

For a Jordan curve $\gamma \subset \mathbb{C}_\infty$, let Ω and Ω^* be the two complementary components of γ in $\mathbb{C}_\infty$.

Conformal weldings

For a Jordan curve $\gamma \subset \mathbb{C}_\infty$, let Ω and Ω^* be the two complementary components of γ in $\mathbb{C}_\infty$.

- By the Riemann mapping theorem, there are conformal maps $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$.

Conformal weldings

For a Jordan curve $\gamma \subset \mathbb{C}_\infty$, let Ω and Ω^* be the two complementary components of γ in $\mathbb{C}_\infty$.

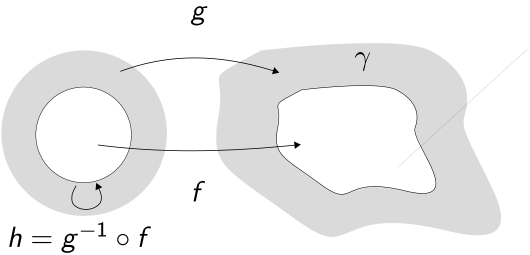
- By the Riemann mapping theorem, there are conformal maps $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$.
- By the Carathéodory theorem, f and g extend to be homeomorphisms of the unit circle to γ.

Conformal weldings

For a Jordan curve $\gamma \subset \mathbb{C}_\infty$, let Ω and Ω^* be the two complementary components of γ in $\mathbb{C}_\infty$.

- By the Riemann mapping theorem, there are conformal maps $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$.
- By the Carathéodory theorem, f and g extend to be homeomorphisms of the unit circle to γ.

The orientation-preserving homeomorphism $h := g^{-1} \circ f : \mathbb{T} \to \mathbb{T}$ is called a conformal welding.



Flexible curves

A Jordan curve γ is said to be flexible if

1. For any Jordan curve γ' and any $\varepsilon > 0$, there exists a homeomorphism $\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty$, conformal off γ, such that $d_H(\phi(\gamma), \gamma') < \varepsilon$, where d_H stands for the Hausdorff distance.

Flexible curves

A Jordan curve γ is said to be flexible if

1. For any Jordan curve γ' and any $\varepsilon > 0$, there exists a homeomorphism $\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty$, conformal off γ, such that $d_H(\phi(\gamma), \gamma') < \varepsilon$, where d_H stands for the Hausdorff distance.
2. Given two points z_1, z_2 in different components of $\mathbb{C}_\infty \setminus \gamma$ and two points w_1, w_2 in different components of $\mathbb{C}_\infty \setminus \gamma'$, ϕ can be taken to satisfy $\phi(z_1) = w_1, \ \phi(z_2) = w_2$.

Logarithmic capacity

- For a finite compactly supported signed Borel measure μ, definne

$$U_\mu(z) = \int \log \frac{1}{|\xi - z|} d\mu(\xi), \quad I(\mu) = \int U_\mu(z) d\mu(z)$$

to be its logarithmic potential and energy integral.

Logarithmic capacity

- For a finite compactly supported signed Borel measure μ, definne

$$U_\mu(z) = \int \log \frac{1}{|\xi - z|} d\mu(\xi), \quad I(\mu) = \int U_\mu(z) d\mu(z)$$

to be its logarithmic potential and energy integral.

- For a compact K, denote by $P(K)$ the set of all Borel probability measures on K and define the Robin's constant of K by

$$\gamma(K) := \inf\{I(\mu) \colon \mu \in P(K)\},$$

and the logarithmic capacity of K by

$$\mathrm{cap}(K) := e^{-\gamma(K)}.$$

Logarithmic capacity

- For a finite compactly supported signed Borel measure μ, definne

$$U_\mu(z) = \int \log \frac{1}{|\xi - z|} d\mu(\xi), \quad I(\mu) = \int U_\mu(z) d\mu(z)$$

  to be its logarithmic potential and energy integral.
- For a compact K, denote by $P(K)$ the set of all Borel probability measures on K and define the Robin's constant of K by

$$\gamma(K) := \inf\{I(\mu) \colon \mu \in P(K)\},$$

  and the logarithmic capacity of K by

$$\mathrm{cap}(K) := e^{-\gamma(K)}.$$

- For a Borel set E,

$$\mathrm{cap}(E) := \sup\{\mathrm{cap}(K) : K \text{ compact}, K \subset E\}.$$

Logarithmic capacity and log-singular homeomorphisms

Lemma 1 (Equivalent definitions of log-singularity)

Let $h : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving circle homeomorphism. Then the following are equivalent:

1. *There is a Borel set E such that*
$$\mathrm{cap}(E) = \mathrm{cap}(h(\mathbb{T} \setminus E)) = 0.$$
2. *For any $\varepsilon > 0$, there is a finite union of closed intervals $E \subset \mathbb{T}$ such that*
$$\mathrm{cap}(E) + \mathrm{cap}(h(\mathbb{T} \setminus E)) < \varepsilon.$$
3. *For any $n \in \mathbb{N}$ there is a compact set $E_n \subset \mathbb{T}$ such that*
$$\mathrm{cap}(E_n) \leq 1/n, \quad \mathrm{cap}(h(\mathbb{T} \setminus E_n)) \leq 1/n.$$

Main theorem

Theorem 1 (Bishop, 2007)

Let h be an orientation-preserving homeomorphism of the circle. Then h is the conformal welding of a flexible curve if and only if it is log-singular.

Necessity

Necessity

1. Take n large enough and let $\Gamma_n := \partial W_n$ with $W := [-1, n^2] \times [-1, 1]$.

Necessity

1. Take n large enough and let $\Gamma_n := \partial W_n$ with $W := [-1, n^2] \times [-1, 1]$.
2. Consider conformal maps $f\colon \mathbb{D} \to \Omega_n$, $g\colon \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \overline{\Omega_n}$ with $f(0) = 0,\ g(\infty) = \infty$, and $d_H(\partial\Omega_n, \Gamma_n) < 1/4$.

Necessity

1. Take n large enough and let $\Gamma_n := \partial W_n$ with $W := [-1, n^2] \times [-1, 1]$.
2. Consider conformal maps $f\colon \mathbb{D} \to \Omega_n$, $g\colon \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \overline{\Omega_n}$ with $f(0) = 0$, $g(\infty) = \infty$, and $d_H(\partial\Omega_n, \Gamma_n) < 1/4$.
3. Let $X_n := \{z \in \Gamma_n : \operatorname{Re} z \geq n\}$, $E_n := f^{-1}(X_n)$, whence $h(\mathbb{T} \setminus E_n) = g^{-1}(\Gamma_n \setminus X_n)$.

Necessity

1. Take n large enough and let $\Gamma_n := \partial W_n$ with $W := [-1, n^2] \times [-1, 1]$.
2. Consider conformal maps $f : \mathbb{D} \to \Omega_n$, $g : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \overline{\Omega_n}$ with $f(0) = 0$, $g(\infty) = \infty$, and $d_H(\partial\Omega_n, \Gamma_n) < 1/4$.
3. Let $X_n := \{z \in \Gamma_n : \mathrm{Re}\, z \geq n\}$, $E_n := f^{-1}(X_n)$, whence $h(\mathbb{T} \setminus E_n) = g^{-1}(\Gamma_n \setminus X_n)$.

Lemma 2 (Balogh & Bonk, 1999)

For $R \geq 1$ and any conformal $f : \mathbb{D} \to \Omega$,

$$\mathrm{cap}(\{x \in \mathbb{T} : |f(x)| \geq R \cdot \mathrm{dist}(f(0), \partial\Omega)\}) \lesssim R^{-1/2}.$$

Necessity

1. Take n large enough and let $\Gamma_n := \partial W_n$ with $W := [-1, n^2] \times [-1, 1]$.
2. Consider conformal maps $f\colon \mathbb{D} \to \Omega_n$, $g\colon \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \overline{\Omega_n}$ with $f(0) = 0$, $g(\infty) = \infty$, and $d_H(\partial\Omega_n, \Gamma_n) < 1/4$.
3. Let $X_n := \{z \in \Gamma_n : \operatorname{Re} z \geq n\}$, $E_n := f^{-1}(X_n)$, whence $h(\mathbb{T} \setminus E_n) = g^{-1}(\Gamma_n \setminus X_n)$.

Lemma 2 (Balogh & Bonk, 1999)

For $R \geq 1$ and any conformal $f : \mathbb{D} \to \Omega$,

$$\operatorname{cap}(\{x \in \mathbb{T} : |f(x)| \geq R \cdot \operatorname{dist}(f(0), \partial\Omega)\}) \lesssim R^{-1/2}.$$

4. For $x \in E_n$, we have $\operatorname{dist}(f(0), \partial\Gamma_n) \approx 1$, $|f(x)| \gtrsim n$, so that $\operatorname{cap}(E_n) \lesssim n^{-1/2}$.

Necessity

1. Take n large enough and let $\Gamma_n := \partial W_n$ with $W := [-1, n^2] \times [-1, 1]$.
2. Consider conformal maps $f : \mathbb{D} \to \Omega_n$, $g : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \overline{\Omega_n}$ with $f(0) = 0$, $g(\infty) = \infty$, and $d_H(\partial\Omega_n, \Gamma_n) < 1/4$.
3. Let $X_n := \{z \in \Gamma_n : \mathrm{Re}\, z \geq n\}$, $E_n := f^{-1}(X_n)$, whence $h(\mathbb{T} \setminus E_n) = g^{-1}(\Gamma_n \setminus X_n)$.

Lemma 2 (Balogh & Bonk, 1999)

For $R \geq 1$ and any conformal $f : \mathbb{D} \to \Omega$,

$$\mathrm{cap}(\{x \in \mathbb{T} : |f(x)| \geq R \cdot \mathrm{dist}(f(0), \partial\Omega)\}) \lesssim R^{-1/2}.$$

4. For $x \in E_n$, we have $\mathrm{dist}(f(0), \partial\Gamma_n) \approx 1$, $|f(x)| \gtrsim n$, so that $\mathrm{cap}(E_n) \lesssim n^{-1/2}$.
5. For $y \in h(\mathbb{T} \setminus E_n)$ and $\tilde{g}(z) := 1/g(1/z)$, we have

$$\mathrm{dist}(\tilde{g}(0), 1/\partial\Gamma_n) \approx 1/n^2, \; |\tilde{g}(y)| \gtrsim 1/n,$$

whence $\mathrm{cap}(h(\mathbb{T} \setminus E_n)) \lesssim n^{-1/2}$.

Sufficiency

Theorem 2

Let $h : \mathbb{T} \to \mathbb{T}$ *be a log-singular orientation-preserving homeomorphism. Assume there are two conformal maps* $F : \mathbb{D} \to \Omega$ *and* $G : \mathbb{D}^* \to \Omega^*$ *with* $\infty \in G(\mathbb{D}^*)$*. Then, for any* $r < 1$ *and* $\eta > 0$*, there are conformal maps* f *and* g *of* $\mathbb{D}$ *and* $\mathbb{D}^*$*, respectively, onto the two complementary components of a Jordan curve* Γ *such that* $h = g^{-1} \circ f$ *on* $\mathbb{T}$ *with*

$$|f(z) - F(z)| \le \eta, \quad |z| \le r,$$
$$|g(z) - G(z)| \le \eta, \quad |z| \ge 1/r.$$

Theorem 2 implies sufficiency

- By Theorem 2, h is a conformal welding corresponding to some γ (with conformal maps f_1 and g_1).

Theorem 2 implies sufficiency

- By Theorem 2, h is a conformal welding corresponding to some γ (with conformal maps f_1 and g_1).
- Given $\varepsilon > 0$, a closed Jordan curve γ', a pair of points z_1, z_2 (w_1, w_2) on either side of γ (γ'), need to show there exists ϕ conformal off γ:

$$d_H(\phi(\gamma), \gamma') < \varepsilon,\ \phi(z_1) = w_1,\ \phi(z_2) = w_2.$$

Theorem 2 implies sufficiency

- By Theorem 2, h is a conformal welding corresponding to some γ (with conformal maps f_1 and g_1).
- Given $\varepsilon > 0$, a closed Jordan curve γ', a pair of points z_1, z_2 (w_1, w_2) on either side of γ (γ'), need to show there exists ϕ conformal off γ:

$$d_H(\phi(\gamma), \gamma') < \varepsilon,\ \phi(z_1) = w_1,\ \phi(z_2) = w_2.$$

- Let F, G be conformal maps of $\mathbb{D}$ and $\mathbb{D}^*$ onto the two sides of γ' mapping $f_1^{-1}(z_1)$ to w_1 and $g_1^{-1}(z_1)$ to w_2.

Theorem 2 implies sufficiency

- By Theorem 2, h is a conformal welding corresponding to some γ (with conformal maps f_1 and g_1).
- Given $\varepsilon > 0$, a closed Jordan curve γ', a pair of points z_1, z_2 (w_1, w_2) on either side of γ (γ'), need to show there exists ϕ conformal off γ:
$$d_H(\phi(\gamma), \gamma') < \varepsilon,\ \phi(z_1) = w_1,\ \phi(z_2) = w_2.$$
- Let F, G be conformal maps of $\mathbb{D}$ and $\mathbb{D}^*$ onto the two sides of γ' mapping $f_1^{-1}(z_1)$ to w_1 and $g_1^{-1}(z_1)$ to w_2.
- Choose $r < 1$ close to 1, so that $F(D(0,1)) \setminus F(D(0,r)) \subset V_{\varepsilon/4}(\gamma')$ (same for G) and assume $|f_1^{-1}(z_1)| < r$, $|g_1^{-1}(z_2)| > 1/r$.

Theorem 2 implies sufficiency

- By Theorem 2, h is a conformal welding corresponding to some γ (with conformal maps f_1 and g_1).
- Given $\varepsilon > 0$, a closed Jordan curve γ', a pair of points z_1, z_2 (w_1, w_2) on either side of γ (γ'), need to show there exists ϕ conformal off γ:
$$d_H(\phi(\gamma), \gamma') < \varepsilon,\ \phi(z_1) = w_1,\ \phi(z_2) = w_2.$$
- Let F, G be conformal maps of $\mathbb{D}$ and $\mathbb{D}^*$ onto the two sides of γ' mapping $f_1^{-1}(z_1)$ to w_1 and $g_1^{-1}(z_1)$ to w_2.
- Choose $r < 1$ close to 1, so that $F(D(0,1)) \setminus F(D(0,r)) \subset V_{\varepsilon/4}(\gamma')$ (same for G) and assume $|f_1^{-1}(z_1)| < r,\ |g_1^{-1}(z_2)| > 1/r$.
- Now, Theorem 2 gives f, g mapping onto two sides of some γ'' satisfying the conditions for $\eta := \frac{\varepsilon}{\max(4, 8(\gamma')+1)|w_1 - w_2|^{-1})}$.

Theorem 2 implies sufficiency

- By Theorem 2, h is a conformal welding corresponding to some γ (with conformal maps f_1 and g_1).
- Given $\varepsilon > 0$, a closed Jordan curve γ', a pair of points z_1, z_2 (w_1, w_2) on either side of γ (γ'), need to show there exists ϕ conformal off γ:

$$d_H(\phi(\gamma), \gamma') < \varepsilon,\ \phi(z_1) = w_1,\ \phi(z_2) = w_2.$$

- Let F, G be conformal maps of $\mathbb{D}$ and $\mathbb{D}^*$ onto the two sides of γ' mapping $f_1^{-1}(z_1)$ to w_1 and $g_1^{-1}(z_1)$ to w_2.
- Choose $r < 1$ close to 1, so that $F(D(0,1)) \setminus F(D(0,r)) \subset V_{\varepsilon/4}(\gamma')$ (same for G) and assume $|f_1^{-1}(z_1)| < r,\ |g_1^{-1}(z_2)| > 1/r$.
- Now, Theorem 2 gives f, g mapping onto two sides of some γ'' satisfying the conditions for $\eta := \frac{\varepsilon}{\max(4, 8(\gamma')+1)|w_1 - w_2|^{-1})}$.
- In particular, $\gamma'' \in V_{\varepsilon/2}(\gamma')$ and $|f(x_1) - w_1|,\ |g(x_2) - w_2| \leq \frac{\varepsilon |w_1 - w_2|}{8 + 8(\gamma')}$.

Theorem 2 implies sufficiency

- By Theorem 2, h is a conformal welding corresponding to some γ (with conformal maps f_1 and g_1).
- Given $\varepsilon > 0$, a closed Jordan curve γ', a pair of points z_1, z_2 (w_1, w_2) on either side of γ (γ'), need to show there exists ϕ conformal off γ:

$$d_H(\phi(\gamma), \gamma') < \varepsilon, \ \phi(z_1) = w_1, \ \phi(z_2) = w_2.$$

- Let F, G be conformal maps of $\mathbb{D}$ and $\mathbb{D}^*$ onto the two sides of γ' mapping $f_1^{-1}(z_1)$ to w_1 and $g_1^{-1}(z_1)$ to w_2.
- Choose $r < 1$ close to 1, so that $F(D(0,1)) \setminus F(D(0,r)) \subset V_{\varepsilon/4}(\gamma')$ (same for G) and assume $|f_1^{-1}(z_1)| < r, \ |g_1^{-1}(z_2)| > 1/r$.
- Now, Theorem 2 gives f, g mapping onto two sides of some γ'' satisfying the conditions for $\eta := \frac{\varepsilon}{\max(4, 8(\gamma')+1)|w_1 - w_2|^{-1})}$.
- In particular, $\gamma'' \in V_{\varepsilon/2}(\gamma')$ and $|f(x_1) - w_1|, \ |g(x_2) - w_2| \leq \frac{\varepsilon |w_1 - w_2|}{8 + 8(\gamma')}$.
- Finally, find a map $s(z) = az + b$ such that $s(f(x_1)) = w_1, \ s(g(x_2)) = w_2$, and $d_H(\gamma'', s(\gamma'')) < \varepsilon/2$.

Proof of Theorem 2

Scheme of the proof. Inductive construction: fix $\varepsilon_0, \delta_0 > 0$.

Proof of Theorem 2

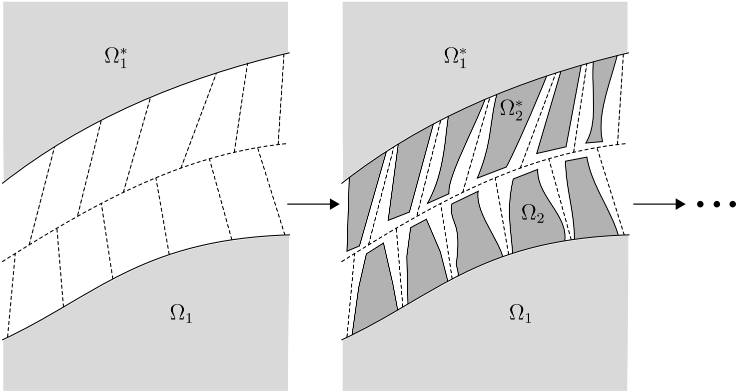
Scheme of the proof. Inductive construction: fix $\varepsilon_0, \delta_0 > 0$.

- Construct K_n-quasiconformal $f_n : \mathbb{D} \to \Omega_n,\ g_n : \mathbb{D}^* \to \Omega^*$, where Ω and Ω^* have disjoint closures and $|K_n - K_{n+1}| \le \varepsilon_0 2^{-n}$.

Proof of Theorem 2

Scheme of the proof. Inductive construction: fix $\varepsilon_0, \delta_0 > 0$.

- Construct K_n-quasiconformal $f_n : \mathbb{D} \to \Omega_n,\ g_n : \mathbb{D}^* \to \Omega^*$, where Ω and Ω^* have disjoint closures and $|K_n - K_{n+1}| \leq \varepsilon_0 2^{-n}$.



Proof of Theorem 2

Scheme of the proof. Inductive construction: fix $\varepsilon_0, \delta_0 > 0$.

- Construct K_n-quasiconformal $f_n : \mathbb{D} \to \Omega_n$, $g_n : \mathbb{D}^* \to \Omega^*$, where Ω and Ω^* have disjoint closures and $|K_n - K_{n+1}| \le \varepsilon_0 2^{-n}$.
- $A_n := \mathbb{C}_\infty \setminus (\Omega \cup \Omega^*)$ is foliated by a family of hyperbolic geodesics $\mathcal{C}_n$: for each $x \in \mathbb{T}$ $\exists!$ $\gamma_n(x) \in \mathcal{C}_n$ which connects $f_n(x) \in \partial\Omega_n$ and $g_n(h(x)) \in \partial\Omega_n^*$ such that for all x,

Proof of Theorem 2

Scheme of the proof. Inductive construction: fix $\varepsilon_0, \delta_0 > 0$.

- Construct K_n-quasiconformal $f_n : \mathbb{D} \to \Omega_n,\ g_n : \mathbb{D}^* \to \Omega^*$, where Ω and Ω^* have disjoint closures and $|K_n - K_{n+1}| \le \varepsilon_0 2^{-n}$.
- $A_n := \mathbb{C}_\infty \setminus (\Omega \cup \Omega^*)$ is foliated by a family of hyperbolic geodesics $\mathcal{C}_n$: for each $x \in \mathbb{T}$ $\exists!$ $\gamma_n(x) \in \mathcal{C}_n$ which connects $f_n(x) \in \partial\Omega_n$ and $g_n(h(x)) \in \partial\Omega_n^*$ such that for all x,
    1. $\ell(\gamma_n(x)) \le 2\ell(\gamma_{n-1}(x)) \Rightarrow |f_n(x) - g_n(h(x))| \le (2/3)^n L$

Proof of Theorem 2

Scheme of the proof. Inductive construction: fix $\varepsilon_0, \delta_0 > 0$.

- Construct K_n-quasiconformal $f_n : \mathbb{D} \to \Omega_n$, $g_n : \mathbb{D}^* \to \Omega^*$, where Ω and Ω^* have disjoint closures and $|K_n - K_{n+1}| \leq \varepsilon_0 2^{-n}$.
- $A_n := \mathbb{C}_\infty \setminus (\Omega \cup \Omega^*)$ is foliated by a family of hyperbolic geodesics $\mathcal{C}_n$: for each $x \in \mathbb{T}$ $\exists!$ $\gamma_n(x) \in \mathcal{C}_n$ which connects $f_n(x) \in \partial\Omega_n$ and $g_n(h(x)) \in \partial\Omega_n^*$ such that for all x,
  1. $\ell(\gamma_n(x)) \leq 2\ell(\gamma_{n-1}(x)) \Rightarrow |f_n(x) - g_n(h(x))| \leq (2/3)^n L$
  2. $|f_n(x) - f_n(h(x))| \leq (2/3)^n L$
  3. $|g_n(x) - g_n(h(x))| \leq (2/3)^n L$

Proof of Theorem 2

Scheme of the proof. Inductive construction: fix $\varepsilon_0, \delta_0 > 0$.

- Construct K_n-quasiconformal $f_n : \mathbb{D} \to \Omega_n,\ g_n : \mathbb{D}^* \to \Omega^*$, where Ω and Ω^* have disjoint closures and $|K_n - K_{n+1}| \leq \varepsilon_0 2^{-n}$.
- $A_n := \mathbb{C}_\infty \setminus (\Omega \cup \Omega^*)$ is foliated by a family of hyperbolic geodesics $\mathcal{C}_n$: for each $x \in \mathbb{T}$ $\exists!$ $\gamma_n(x) \in \mathcal{C}_n$ which connects $f_n(x) \in \partial\Omega_n$ and $g_n(h(x)) \in \partial\Omega_n^*$ such that for all x,
  1. $\ell(\gamma_n(x)) \leq 2\ell(\gamma_{n-1}(x)) \Rightarrow |f_n(x) - g_n(h(x))| \leq (2/3)^n L$
  2. $|f_n(x) - f_n(h(x))| \leq (2/3)^n L$
  3. $|g_n(x) - g_n(h(x))| \leq (2/3)^n L$
  4. $|F(x) - f_0(x)| \leq \delta_0, \quad |z| \leq 1 - \delta_0,$
  5. $|G(x) - g_0(x)| \leq \delta_0, \quad |z| \geq 1 + \delta_0.$

Proof of Theorem 2

Scheme of the proof. Inductive construction: fix $\varepsilon_0, \delta_0 > 0$.

- Construct K_n-quasiconformal $f_n : \mathbb{D} \to \Omega_n,\ g_n : \mathbb{D}^* \to \Omega^*$, where Ω and Ω^* have disjoint closures and $|K_n - K_{n+1}| \leq \varepsilon_0 2^{-n}$.
- $A_n := \mathbb{C}_\infty \setminus (\Omega \cup \Omega^*)$ is foliated by a family of hyperbolic geodesics $\mathcal{C}_n$: for each $x \in \mathbb{T}\ \exists!\ \gamma_n(x) \in \mathcal{C}_n$ which connects $f_n(x) \in \partial\Omega_n$ and $g_n(h(x)) \in \partial\Omega_n^*$ such that for all x,
  1. $\ell(\gamma_n(x)) \leq 2\ell(\gamma_{n-1}(x)) \Rightarrow |f_n(x) - g_n(h(x))| \leq (2/3)^n L$
  2. $|f_n(x) - f_n(h(x))| \leq (2/3)^n L$
  3. $|g_n(x) - g_n(h(x))| \leq (2/3)^n L$
  4. $|F(x) - f_0(x)| \leq \delta_0, \quad |z| \leq 1 - \delta_0,$
  5. $|G(x) - g_0(x)| \leq \delta_0, \quad |z| \geq 1 + \delta_0.$
- Then by the Weierstrass theorem, items 2 and 3 imply that $f_n \to f \in C(\mathbb{D})$ and $g_n \to g \in C(\mathbb{D}^*)$ uniformly (locally connected boundaries)

Proof of Theorem 2

Scheme of the proof. Inductive construction: fix $\varepsilon_0, \delta_0 > 0$.

- Construct K_n-quasiconformal $f_n : \mathbb{D} \to \Omega_n$, $g_n : \mathbb{D}^* \to \Omega^*$, where Ω and Ω^* have disjoint closures and $|K_n - K_{n+1}| \leq \varepsilon_0 2^{-n}$.
- $A_n := \mathbb{C}_\infty \setminus (\Omega \cup \Omega^*)$ is foliated by a family of hyperbolic geodesics $\mathcal{C}_n$: for each $x \in \mathbb{T}$ $\exists!$ $\gamma_n(x) \in \mathcal{C}_n$ which connects $f_n(x) \in \partial\Omega_n$ and $g_n(h(x)) \in \partial\Omega_n^*$ such that for all x,
  1. $\ell(\gamma_n(x)) \leq 2\ell(\gamma_{n-1}(x)) \Rightarrow |f_n(x) - g_n(h(x))| \leq (2/3)^n L$
  2. $|f_n(x) - f_n(h(x))| \leq (2/3)^n L$
  3. $|g_n(x) - g_n(h(x))| \leq (2/3)^n L$
  4. $|F(x) - f_0(x)| \leq \delta_0, \quad |z| \leq 1 - \delta_0,$
  5. $|G(x) - g_0(x)| \leq \delta_0, \quad |z| \geq 1 + \delta_0.$
- Then by the Weierstrass theorem, items 2 and 3 imply that $f_n \to f \in C(\mathbb{D})$ and $g_n \to g \in C(\mathbb{D}^*)$ uniformly (locally connected boundaries)
- By item 1, $f = g \circ h$, so that f and g are one-to-one on $\mathbb{T}$ and therefore are maps onto two sides of a Jordan curve.

Proof of Theorem 2

Scheme of the proof. Inductive construction: fix $\varepsilon_0, \delta_0 > 0$.

- Construct K_n-quasiconformal $f_n : \mathbb{D} \to \Omega_n$, $g_n : \mathbb{D}^* \to \Omega^*$, where Ω and Ω^* have disjoint closures and $|K_n - K_{n+1}| \leq \varepsilon_0 2^{-n}$.
- $A_n := \mathbb{C}_\infty \setminus (\Omega \cup \Omega^*)$ is foliated by a family of hyperbolic geodesics $\mathcal{C}_n$: for each $x \in \mathbb{T}$ $\exists!$ $\gamma_n(x) \in \mathcal{C}_n$ which connects $f_n(x) \in \partial\Omega_n$ and $g_n(h(x)) \in \partial\Omega_n^*$ such that for all x,
  1. $\ell(\gamma_n(x)) \leq 2\ell(\gamma_{n-1}(x)) \Rightarrow |f_n(x) - g_n(h(x))| \leq (2/3)^n L$
  2. $|f_n(x) - f_n(h(x))| \leq (2/3)^n L$
  3. $|g_n(x) - g_n(h(x))| \leq (2/3)^n L$
  4. $|F(x) - f_0(x)| \leq \delta_0, \quad |z| \leq 1 - \delta_0,$
  5. $|G(x) - g_0(x)| \leq \delta_0, \quad |z| \geq 1 + \delta_0.$
- Then by the Weierstrass theorem, items 2 and 3 imply that $f_n \to f \in C(\mathbb{D})$ and $g_n \to g \in C(\mathbb{D}^*)$ uniformly (locally connected boundaries)
- By item 1, $f = g \circ h$, so that f and g are one-to-one on $\mathbb{T}$ and therefore are maps onto two sides of a Jordan curve.
- Items 2-5 show that f and g approximate F and G on compact sets.

Construction of f_n's

We want to construct $f_{n+1} := \Psi \circ \Phi$, where:

Construction of f_n's

We want to construct $f_{n+1} := \Psi \circ \Phi$, where:

- $\Phi : \mathbb{D} \to W$, where W is like a disk with large radius $R \approx \exp(A/N + \mathcal{O}(N^{-1} \log N))$ and finitely many radial slits removed.

Construction of f_n's

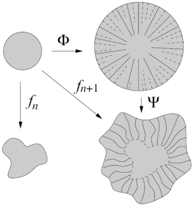
We want to construct $f_{n+1} := \Psi \circ \Phi$, where:

- $\Phi : \mathbb{D} \to W$, where W is like a disk with large radius $R \approx \exp(A/N + \mathcal{O}(N^{-1} \log N))$ and finitely many radial slits removed. The set E goes to the boundary of the "disk" and $\mathbb{T} \setminus E$, to the slits.

Construction of f_n's

We want to construct $f_{n+1} := \Psi \circ \Phi$, where:

- $\Phi : \mathbb{D} \to W$, where W is like a disk with large radius $R \approx \exp(A/N + \mathcal{O}(N^{-1} \log N))$ and finitely many radial slits removed. The set E goes to the boundary of the "disk" and $\mathbb{T} \setminus E$, to the slits.
- Ψ is a quasiconformal (with constant close to 1) extension of f_n from $\mathbb{D}$ to $V \supset W$.



Auxiliary map Φ

We want $\Phi = \exp(U + i\tilde{U})$ with $U(z) = G(z) + G(1/\overline{z})$, where G is a potential of a measure supported on $E \in \mathbb{T}$.

Auxiliary map Φ

We want $\Phi = \exp(U + i\tilde{U})$ with $U(z) = G(z) + G(1/\overline{z})$, where G is a potential of a measure supported on $E \in \mathbb{T}$.

- Take N large (to be chosen later) and divide $\mathbb{T}$ into N equal intervals (I_k).

Auxiliary map Φ

We want $\Phi = \exp(U + i\tilde{U})$ with $U(z) = G(z) + G(1/\overline{z})$, where G is a potential of a measure supported on $E \in \mathbb{T}$.

- Take N large (to be chosen later) and divide $\mathbb{T}$ into N equal intervals (I_k).
- Given $\varepsilon_1 > 0$, choose $E \subset \mathbb{T}$ to be a finite union of intervals such that

$$\mathrm{cap}(E_k) := \mathrm{cap}(I_k \cap E) < \varepsilon_1, \ \mathrm{cap}(F_k) := \mathrm{cap}(h(I_k \setminus E)) < \varepsilon_1.$$

Auxiliary map Φ

We want $\Phi = \exp(U + i\tilde{U})$ with $U(z) = G(z) + G(1/\bar{z})$, where G is a potential of a measure supported on $E \in \mathbb{T}$.

- Take N large (to be chosen later) and divide $\mathbb{T}$ into N equal intervals (I_k).
- Given $\varepsilon_1 > 0$, choose $E \subset \mathbb{T}$ to be a finite union of intervals such that

$$\mathrm{cap}(E_k) := \mathrm{cap}(I_k \cap E) < \varepsilon_1, \; \mathrm{cap}(F_k) := \mathrm{cap}(h(I_k \setminus E)) < \varepsilon_1.$$

- We can make the Robin constant of E_k and F_k equal to A (large, to be chosen later) for small enough ε_1.

Auxiliary map Φ

We want $\Phi = \exp(U + i\tilde{U})$ with $U(z) = G(z) + G(1/\bar{z})$, where G is a potential of a measure supported on $E \in \mathbb{T}$.

- Take N large (to be chosen later) and divide $\mathbb{T}$ into N equal intervals (I_k).
- Given $\varepsilon_1 > 0$, choose $E \subset \mathbb{T}$ to be a finite union of intervals such that

$$\operatorname{cap}(E_k) := \operatorname{cap}(I_k \cap E) < \varepsilon_1, \ \operatorname{cap}(F_k) := \operatorname{cap}(h(I_k \setminus E)) < \varepsilon_1.$$

- We can make the Robin constant of E_k and F_k equal to A (large, to be chosen later) for small enough ε_1.
- Put the equilibrium measure μ_k of mass N on $E_k := I_k \cap E$ and let $\mu = \sum_k \mu_k$.

Properties of the potential G

Let

$$G(z) := \int \log \frac{1}{|x-z|} d\mu(x) =: \sum_k G_k(z).$$

Properties of the potential G

Let

$$G(z) := \int \log \frac{1}{|x-z|} d\mu(x) =: \sum_k G_k(z).$$

The function $NG_k - A$ is the Green function for $\Omega_k := \mathbb{C}_\infty \setminus E_k$ with pole at ∞ (and hence is zero on E_k).

Properties of the potential G

Let

$$G(z) := \int \log \frac{1}{|x-z|} d\mu(x) =: \sum_k G_k(z).$$

The function $NG_k - A$ is the Green function for $\Omega_k := \mathbb{C}_\infty \setminus E_k$ with pole at ∞ (and hence is zero on E_k). Thus, $G_k(z) = A/N$ for $z \in E_k$.

Properties of the potential G

Let

$$G(z) := \int \log \frac{1}{|x-z|} d\mu(x) =: \sum_k G_k(z).$$

The function $NG_k - A$ is the Green function for $\Omega_k := \mathbb{C}_\infty \setminus E_k$ with pole at ∞ (and hence is zero on E_k). Thus, $G_k(z) = A/N$ for $z \in E_k$.

Lemma 1

G is continuous on $\mathbb{C}$ and and harmonic on $\mathbb{C} \setminus E$. Moreover,

1. $G(z) \to \log |z|^{-1}$, $|z| > 1$, *and* $G(z) \to 0$, $|z| < 1$, *as* $N \to \infty$.
2. *For any* $\delta > 0$ *and any interval* I, $\frac{|\{x \in I : |G(x)| > \delta\}|}{|I|} \to 0$ *as* $N \to \infty$.

Properties of the potential G: proof

Item 1 follows from the fact that $\mu \rightharpoonup$ normalized Lebesgue and that

$$\int \log|z-x|^{-1} dx = \begin{cases} 0, & |z| < 1, \\ \log|z|^{-1}, & |z| > 1. \end{cases}$$

Properties of the potential G: proof

Item 1 follows from the fact that $\mu \rightharpoonup$ normalized Lebesgue and that

$$\int \log|z-x|^{-1} dx = \begin{cases} 0, & |z| < 1, \\ \log|z|^{-1}, & |z| > 1. \end{cases}$$

For item 2, relabel I_k's so that I_1, I_2 are adjacent to $I_0 \ni x$ and $\operatorname{dist}(I_k, x) \approx k/N$ for $k \geq 3$. Also, let $I = I_0 \cap I_1 \cap I_2$.

Properties of the potential G: proof

Item 1 follows from the fact that $\mu \rightharpoonup$ normalized Lebesgue and that

$$\int \log |z - x|^{-1} dx = \begin{cases} 0, & |z| < 1, \\ \log |z|^{-1}, & |z| > 1. \end{cases}$$

For item 2, relabel I_k's so that I_1, I_2 are adjacent to $I_0 \ni x$ and $\mathrm{dist}(I_k, x) \approx k/N$ for $k \geq 3$. Also, let $I = I_0 \cap I_1 \cap I_2$.
Let

$$G(x) = \Big(\int_I + \sum_{k \geq 3} \int_{I_k} \Big) \log |z - x|^{-1} d\mu(z) =: H_1(x) + H_2(x).$$

Properties of the potential G: proof

Item 1 follows from the fact that $\mu \rightharpoonup$ normalized Lebesgue and that

$$\int \log|z-x|^{-1}dx = \begin{cases} 0, & |z|<1, \\ \log|z|^{-1}, & |z|>1. \end{cases}$$

For item 2, relabel I_k's so that I_1, I_2 are adjacent to $I_0 \ni x$ and $\operatorname{dist}(I_k, x) \approx k/N$ for $k \geq 3$. Also, let $I = I_0 \cap I_1 \cap I_2$.

Let

$$G(x) = \Big(\int_I + \sum_{k\geq 3} \int_{I_k} \Big) \log|z-x|^{-1} d\mu(z) =: H_1(x) + H_2(x).$$

For $N \geq 12$, we have $H_1 > 0$ and

$$\int_I H_1(x)dx \leq \mu(I) \max_{z\in I} \int_I \log|z-x|^{-1}|dx| \lesssim \int_0^{N^{-1}} y \ln y^{-1} dy \lesssim \frac{\log N}{N^2}.$$

Properties of the potential G: proof

Item 1 follows from the fact that $\mu \rightharpoonup$ normalized Lebesgue and that

$$\int \log|z-x|^{-1}dx = \begin{cases} 0, & |z|<1, \\ \log|z|^{-1}, & |z|>1. \end{cases}$$

For item 2, relabel I_k's so that I_1, I_2 are adjacent to $I_0 \ni x$ and $\mathrm{dist}(I_k, x) \approx k/N$ for $k \geq 3$. Also, let $I = I_0 \cap I_1 \cap I_2$.
Let

$$G(x) = \Big(\int_I + \sum_{k\geq 3}\int_{I_k}\Big)\log|z-x|^{-1}d\mu(z) =: H_1(x) + H_2(x).$$

For $N \geq 12$, we have $H_1 > 0$ and

$$\int_I H_1(x)dx \leq \mu(I)\max_{z\in I}\int_I \log|z-x|^{-1}|dx| \lesssim \int_0^{N^{-1}} y\ln y^{-1}dy \lesssim \frac{\log N}{N^2}.$$

By Chebyshev, $|\{x \in I_0 : H_1 \geq \frac{\lambda}{2}\frac{\log N}{N}\}| \lesssim \frac{1}{N\lambda}$, whence for $\lambda := 2\delta N/\log N$,

$$|\{x \in I_0 : H_1 \geq \delta\}| \lesssim \frac{\log N}{2N^2\delta}.$$

Properties of the potential G: proof

Next, since for $k \geq 3$, $\mathrm{Var}_{I_k} \log|z-x|^{-1} \lesssim k^{-1}$,

$$
\begin{aligned}
|H_2(x)| \leq & \Big| \sum_{k\geq 3} \int_{I_k} \log|z-x|^{-1} \Big(d\mu(z) - \frac{|dz|}{2\pi} \Big) \Big| \\
& + \Big| \sum_{k\geq 3} \int_{I_k} \log|z-x|^{-1} \frac{|dz|}{2\pi} \Big| \\
& \lesssim \sum_{k\geq 3} \frac{1}{Nk} + \int_I \log|z-x|^{-1} \frac{|dz|}{2\pi} \lesssim \frac{\log N}{N}.
\end{aligned}
$$

Properties of the potential G: proof

Next, since for $k \geq 3$, $\mathrm{Var}_{I_k} \log|z-x|^{-1} \lesssim k^{-1}$,

$$
\begin{aligned}
|H_2(x)| &\leq \Big| \sum_{k\geq 3} \int_{I_k} \log|z-x|^{-1} \Big(d\mu(z) - \frac{|dz|}{2\pi} \Big) \Big| \\
&+ \Big| \sum_{k\geq 3} \int_{I_k} \log|z-x|^{-1} \frac{|dz|}{2\pi} \Big| \\
&\lesssim \sum_{k\geq 3} \frac{1}{Nk} + \int_I \log|z-x|^{-1} \frac{|dz|}{2\pi} \lesssim \frac{\log N}{N}.
\end{aligned}
$$

- Thus, for $x \in E$, we have $G(x) \geq A/N + \mathcal{O}(\log N/N)$.

Finish the construction of Φ

- Symmetrize: $U(z) := G(z) + G(z/|z|^2)$, so that U is harmonic on Ω, $U(x) = 2G(x)$ on E and has negative logarithmic poles at 0 and ∞.

Finish the construction of Φ

- Symmetrize: $U(z) := G(z) + G(z/|z|^2)$, so that U is harmonic on Ω, $U(x) = 2G(x)$ on E and has negative logarithmic poles at 0 and ∞.
- A (multi-valued) harmonic conjugate $\tilde{U}$ of U on $\mathbb{D}$ can be chosen so that $\exp(i\tilde{U}(x)) \to x$ uniformly as $N \to \infty$ (independently of A).

Finish the construction of Φ

- Symmetrize: $U(z) := G(z) + G(z/|z|^2)$, so that U is harmonic on Ω, $U(x) = 2G(x)$ on E and has negative logarithmic poles at 0 and ∞.
- A (multi-valued) harmonic conjugate $\tilde{U}$ of U on $\mathbb{D}$ can be chosen so that $\exp(i\tilde{U}(x)) \to x$ uniformly as $N \to \infty$ (independently of A).
- $\varphi(z) := \exp(U(z) + i\tilde{U}(z))$ is a conformal map of $\mathbb{D}$ onto a region W which is a Jordan W' with a finite number of radial slits removed.

Finish the construction of Φ

- Symmetrize: $U(z) := G(z) + G(z/|z|^2)$, so that U is harmonic on Ω, $U(x) = 2G(x)$ on E and has negative logarithmic poles at 0 and ∞.
- A (multi-valued) harmonic conjugate $\tilde{U}$ of U on $\mathbb{D}$ can be chosen so that $\exp(i\tilde{U}(x)) \to x$ uniformly as $N \to \infty$ (independently of A).
- $\varphi(z) := \exp(U(z) + i\tilde{U}(z))$ is a conformal map of $\mathbb{D}$ onto a region W which is a Jordan W' with a finite number of radial slits removed.
- Since $G(x) \geq A/N + \mathcal{O}(\log N/N)$, $D(0, \exp((2A/N - 1)) \subset W'$ for N large enough and $D(0, 1 - C \log N/N) \subset W$ for some C.

Finish the construction of Φ

- Symmetrize: $U(z) := G(z) + G(z/|z|^2)$, so that U is harmonic on Ω, $U(x) = 2G(x)$ on E and has negative logarithmic poles at 0 and ∞.
- A (multi-valued) harmonic conjugate $\tilde{U}$ of U on $\mathbb{D}$ can be chosen so that $\exp(i\tilde{U}(x)) \to x$ uniformly as $N \to \infty$ (independently of A).
- $\varphi(z) := \exp(U(z) + i\tilde{U}(z))$ is a conformal map of $\mathbb{D}$ onto a region W which is a Jordan W' with a finite number of radial slits removed.
- Since $G(x) \geq A/N + \mathcal{O}(\log N/N)$, $D(0, \exp((2A/N - 1)) \subset W'$ for N large enough and $D(0, 1 - C\log N/N) \subset W$ for some C.
- On the other hand, any interval of length $1/N$ contains a point x with $U(x) < C\log N/N$ and hence any such interval contains a point with $|\varphi(x)| \leq 1 + C\log N/N$.

Finish the construction of Φ

- Symmetrize: $U(z) := G(z) + G(z/|z|^2)$, so that U is harmonic on Ω, $U(x) = 2G(x)$ on E and has negative logarithmic poles at 0 and ∞.
- A (multi-valued) harmonic conjugate $\tilde{U}$ of U on $\mathbb{D}$ can be chosen so that $\exp(i\tilde{U}(x)) \to x$ uniformly as $N \to \infty$ (independently of A).
- $\varphi(z) := \exp(U(z) + i\tilde{U}(z))$ is a conformal map of $\mathbb{D}$ onto a region W which is a Jordan W' with a finite number of radial slits removed.
- Since $G(x) \geq A/N + \mathcal{O}(\log N/N)$, $D(0, \exp((2A/N - 1)) \subset W'$ for N large enough and $D(0, 1 - C\log N/N) \subset W$ for some C.
- On the other hand, any interval of length $1/N$ contains a point x with $U(x) < C\log N/N$ and hence any such interval contains a point with $|\varphi(x)| \leq 1 + C\log N/N$.
- Let $M \in \mathbb{N}$ be large and $\delta > 0$ be small. If A and N are large enough, we can choose M critical points (x_k) of G on $\mathbb{T}$ so that $|x_k - \exp(2\pi i/M)| \leq \delta$ and $1 - \delta \leq \varphi(x_k) \leq 1 + \delta$ for all $1 \leq k \leq M$.

Finish the construction of Φ

- Symmetrize: $U(z) := G(z) + G(z/|z|^2)$, so that U is harmonic on Ω, $U(x) = 2G(x)$ on E and has negative logarithmic poles at 0 and ∞.
- A (multi-valued) harmonic conjugate $\tilde{U}$ of U on $\mathbb{D}$ can be chosen so that $\exp(i\tilde{U}(x)) \to x$ uniformly as $N \to \infty$ (independently of A).
- $\varphi(z) := \exp(U(z) + i\tilde{U}(z))$ is a conformal map of $\mathbb{D}$ onto a region W which is a Jordan W' with a finite number of radial slits removed.
- Since $G(x) \geq A/N + \mathcal{O}(\log N/N)$, $D(0, \exp((2A/N - 1)) \subset W'$ for N large enough and $D(0, 1 - C\log N/N) \subset W$ for some C.
- On the other hand, any interval of length $1/N$ contains a point x with $U(x) < C\log N/N$ and hence any such interval contains a point with $|\varphi(x)| \leq 1 + C\log N/N$.
- Let $M \in \mathbb{N}$ be large and $\delta > 0$ be small. If A and N are large enough, we can choose M critical points (x_k) of G on $\mathbb{T}$ so that $|x_k - \exp(2\pi i/M)| \leq \delta$ and $1 - \delta \leq \varphi(x_k) \leq 1 + \delta$ for all $1 \leq k \leq M$.
- Let $\Phi := \varphi/(1 + \delta)$.

Construction of Ψ

- Let W_k be the interior of the closure of the part of $W := \Phi(\mathbb{D})$ separated from 0 by the arc of $\mathbb{T}$ between x_k and x_{k+1}.

Construction of Ψ

- Let W_k be the interior of the closure of the part of $W := \Phi(\mathbb{D})$ separated from 0 by the arc of $\mathbb{T}$ between x_k and x_{k+1}.
- W_k can be conformally mapped to $\mathcal{R}_k := \{z \in \mathbb{C} : \operatorname{Im} z \in (0,1),\ \operatorname{Re} z \in (0, R_k)\}$ with x_k and x_{k+1} going to 0 and i and the radial sides of W_k going to the horizontal sides of $\mathcal{R}_k$.

Construction of Ψ

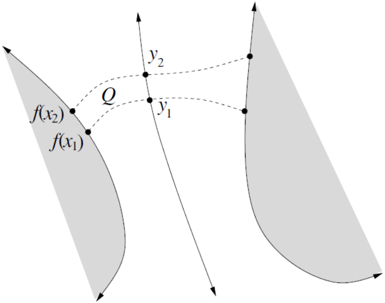
- Let W_k be the interior of the closure of the part of $W := \Phi(\mathbb{D})$ separated from 0 by the arc of $\mathbb{T}$ between x_k and x_{k+1}.
- W_k can be conformally mapped to $\mathcal{R}_k := \{z \in \mathbb{C} : \operatorname{Im} z \in (0,1),\ \operatorname{Re} z \in (0, R_k)\}$ with x_k and x_{k+1} going to 0 and i and the radial sides of W_k going to the horizontal sides of $\mathcal{R}_k$.
- For any large R, we can take $R_k \geq R$ for all k if $\delta > 0$ is small enough and A, N are large enough.

Construction of Ψ

- Consider the set of midpoints of $\gamma \in \mathcal{C}_n$, which must form a smooth curve Γ_n separating the boundary components of $A_n = \mathbb{C}_\infty \setminus (\Omega \cup \Omega^*)$.

Construction of Ψ

- Consider the set of midpoints of $\gamma \in \mathcal{C}_n$, which must form a smooth curve Γ_n separating the boundary components of $A_n = \mathbb{C}_\infty \setminus (\Omega \cup \Omega^*)$.
- For any $x_1, x_2 \in \mathbb{T}$ consider the topological quadrilateral $Q := Q(x_1, x_2)$:



Construction of Ψ

Given $\eta > 0$, assume x_1, x_2 are sufficiently close, so that

Construction of Ψ

Given $\eta > 0$, assume x_1, x_2 are sufficiently close, so that

1. Q is a $1 + 2^{-n}\varepsilon_0$ image of a rectangle,

Construction of Ψ

Given $\eta > 0$, assume x_1, x_2 are sufficiently close, so that

1. Q is a $1 + 2^{-n}\varepsilon_0$ image of a rectangle,
2. The side of $Q(x_1, x_2)$ along Γ_n has length $\leq \eta$,

Construction of Ψ

Given $\eta > 0$, assume x_1, x_2 are sufficiently close, so that

1. Q is a $1 + 2^{-n}\varepsilon_0$ image of a rectangle,
2. The side of $Q(x_1, x_2)$ along Γ_n has length $\leq \eta$,
3. $Q(x_1, x_2)$ can be conformally mapped to $\mathcal{T}(x_1, x_2) := \{z \in \mathbb{C} : \mathrm{Re}\, z \in (0, T(x_1, x_2)),\ \mathrm{Im}\, z \in (0, 1)\}$ (where $T(x_1, x_2) \to \infty$ as $|x_1 - x_2| \to 0$).

Construction of Ψ

Given $\eta > 0$, assume x_1, x_2 are sufficiently close, so that

1. Q is a $1 + 2^{-n}\varepsilon_0$ image of a rectangle,
2. The side of $Q(x_1, x_2)$ along Γ_n has length $\leq \eta$,
3. $Q(x_1, x_2)$ can be conformally mapped to $\mathcal{T}(x_1, x_2) := \{z \in \mathbb{C} : \mathrm{Re}\, z \in (0, T(x_1, x_2)),\ \mathrm{Im}\, z \in (0, 1)\}$ (where $T(x_1, x_2) \to \infty$ as $|x_1 - x_2| \to 0$).

Define $T^*(x_1, x_2)$ similarly and let

$$T(\delta) := \max\{T(x_1, x_2) + T^*(x_1, x_2) : |x_1 - x_2| \geq \delta\}.$$

Construction of Ψ

Given $\eta > 0$, assume x_1, x_2 are sufficiently close, so that

1. Q is a $1 + 2^{-n}\varepsilon_0$ image of a rectangle,
2. The side of $Q(x_1, x_2)$ along Γ_n has length $\leq \eta$,
3. $Q(x_1, x_2)$ can be conformally mapped to $\mathcal{T}(x_1, x_2) := \{z \in \mathbb{C} : \mathrm{Re}\, z \in (0, T(x_1, x_2)),\ \mathrm{Im}\, z \in (0, 1)\}$ (where $T(x_1, x_2) \to \infty$ as $|x_1 - x_2| \to 0$).

Define $T^*(x_1, x_2)$ similarly and let

$$T(\delta) := \max\{T(x_1, x_2) + T^*(x_1, x_2) : |x_1 - x_2| \geq \delta\}.$$

Choose A, N so that $R_k > T(\frac{1}{2M})$ for all k.

Construction of Ψ

- Since $R_k \geq T_k := T(x_k, x_{k+1})$, map $\mathcal{R}_k$ into $\mathcal{T}(x_k, x_{k+1})$: "left" side bijectively to the left, "right" side into (not necessarily onto), the right side.

Construction of Ψ

- Since $R_k \geq T_k := T(x_k, x_{k+1})$, map $\mathcal{R}_k$ into $\mathcal{T}(x_k, x_{k+1})$: "left" side bijectively to the left, "right" side into (not necessarily onto), the right side.
- Map it to Q_k by a conformal map.

Construction of Ψ

- Since $R_k \geq T_k := T(x_k, x_{k+1})$, map $\mathcal{R}_k$ into $\mathcal{T}(x_k, x_{k+1})$: "left" side bijectively to the left, "right" side into (not necessarily onto), the right side.
- Map it to Q_k by a conformal map.
- The kth radial sector D_k is mapped now to the kth quadrilateral.

Construction of Ψ

- Since $R_k \geq T_k := T(x_k, x_{k+1})$, map $\mathcal{R}_k$ into $\mathcal{T}(x_k, x_{k+1})$: "left" side bijectively to the left, "right" side into (not necessarily onto), the right side.
- Map it to Q_k by a conformal map.
- The kth radial sector D_k is mapped now to the kth quadrilateral.
- If the map is quasiconformal with constant $1 + 2^{-n}\varepsilon_0$, we can glue it to $f_n : \mathbb{D} \to \Omega_n$ so that every point of E is mapped into Γ_n (mapped to within η of the point where $\gamma(x)$ crosses Γ_n).

Conclusion

- Repeat the process for Ω^* to get a $(1 + 2^{-n}\varepsilon_0)$-quasiconformal map g_{n+1} that maps each point of $h(\mathbb{T} \setminus E)$ to within η of the points where $\gamma(x)$ crosses Γ_n.

Conclusion

- Repeat the process for Ω^* to get a $(1 + 2^{-n}\varepsilon_0)$-quasiconformal map g_{n+1} that maps each point of $h(\mathbb{T} \setminus E)$ to within η of the points where $\gamma(x)$ crosses Γ_n.
- For all $x \in \mathbb{T}$, $f_{n+1}(x)$ and $g_{n+1}(x)$ can be joined by a curve of length $\leq \ell(\gamma_n(x))/2 + C\varepsilon$ in $\mathbb{C}_\infty \setminus (f_{n+1}(\mathbb{D}) \cup g_{n+1}(\mathbb{D}))$.

Conclusion

- Repeat the process for Ω^* to get a $(1 + 2^{-n}\varepsilon_0)$-quasiconformal map g_{n+1} that maps each point of $h(\mathbb{T} \setminus E)$ to within η of the points where $\gamma(x)$ crosses Γ_n.
- For all $x \in \mathbb{T}$, $f_{n+1}(x)$ and $g_{n+1}(x)$ can be joined by a curve of length $\leq \ell(\gamma_n(x))/2 + C\varepsilon$ in $\mathbb{C}_\infty \setminus (f_{n+1}(\mathbb{D}) \cup g_{n+1}(\mathbb{D}))$.
- Replace $f_{n+1}(z)$ and $g_{n+1}(z)$ by $f_{n+1}(tz)$ and $g_{n+1}(z/t)$ for some $t < 1$, so that the maps are onto smooth disjoint Jordan domains.

Conclusion

- Repeat the process for Ω^* to get a $(1+2^{-n}\varepsilon_0)$-quasiconformal map g_{n+1} that maps each point of $h(\mathbb{T}\setminus E)$ to within η of the points where $\gamma(x)$ crosses Γ_n.
- For all $x \in \mathbb{T}$, $f_{n+1}(x)$ and $g_{n+1}(x)$ can be joined by a curve of length $\leq \ell(\gamma_n(x))/2 + C\varepsilon$ in $\mathbb{C}_\infty \setminus (f_{n+1}(\mathbb{D}) \cup g_{n+1}(\mathbb{D}))$.
- Replace $f_{n+1}(z)$ and $g_{n+1}(z)$ by $f_{n+1}(tz)$ and $g_{n+1}(z/t)$ for some $t<1$, so that the maps are onto smooth disjoint Jordan domains.
- Let $\Omega_{n+1} := f_{n+1}(t\mathbb{D})$ and $\Omega^*_{n+1} := g_{n+1}(\mathbb{D}^*/t)$.

Conclusion

- Repeat the process for Ω^* to get a $(1+2^{-n}\varepsilon_0)$-quasiconformal map g_{n+1} that maps each point of $h(\mathbb{T} \setminus E)$ to within η of the points where $\gamma(x)$ crosses Γ_n.
- For all $x \in \mathbb{T}$, $f_{n+1}(x)$ and $g_{n+1}(x)$ can be joined by a curve of length $\leq \ell(\gamma_n(x))/2 + C\varepsilon$ in $\mathbb{C}_\infty \setminus (f_{n+1}(\mathbb{D}) \cup g_{n+1}(\mathbb{D}))$.
- Replace $f_{n+1}(z)$ and $g_{n+1}(z)$ by $f_{n+1}(tz)$ and $g_{n+1}(z/t)$ for some $t < 1$, so that the maps are onto smooth disjoint Jordan domains.
- Let $\Omega_{n+1} := f_{n+1}(t\mathbb{D})$ and $\Omega^*_{n+1} := g_{n+1}(\mathbb{D}^*/t)$.
- Join $f_{n+1}(tx)$ to $g_{n+1}(h(x)/t)$ by the hyperbolic geodesic in A_{n+1}.

Conclusion

- Repeat the process for Ω^* to get a $(1+2^{-n}\varepsilon_0)$-quasiconformal map g_{n+1} that maps each point of $h(\mathbb{T}\setminus E)$ to within η of the points where $\gamma(x)$ crosses Γ_n.
- For all $x\in\mathbb{T}$, $f_{n+1}(x)$ and $g_{n+1}(x)$ can be joined by a curve of length $\leq \ell(\gamma_n(x))/2 + C\varepsilon$ in $\mathbb{C}_\infty \setminus (f_{n+1}(\mathbb{D})\cup g_{n+1}(\mathbb{D}))$.
- Replace $f_{n+1}(z)$ and $g_{n+1}(z)$ by $f_{n+1}(tz)$ and $g_{n+1}(z/t)$ for some $t<1$, so that the maps are onto smooth disjoint Jordan domains.
- Let $\Omega_{n+1} := f_{n+1}(t\mathbb{D})$ and $\Omega^*_{n+1} := g_{n+1}(\mathbb{D}^*/t)$.
- Join $f_{n+1}(tx)$ to $g_{n+1}(h(x)/t)$ by the hyperbolic geodesic in A_{n+1}.
- This geodesic has length $\leq \ell(\gamma_n(x))/2 + C\varepsilon$ (black box).

Thank you for your attention!