A lecture on Conformal Uniformization of Domains Bounded by Quasitripods

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Conformal uniformization

Riemann mapping theorem

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then there exists a conformal map $f : \mathbb{D} \to \Omega$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

What if Ω is not simply connected? What should replace the unit disk as the uniformizing domain?

Circle domain uniformization	**Parallel slit uniformization**
Koebe (1908) conjectured that every domain in the Riemann sphere $\widehat{\mathbb{C}}$ is conformally equivalent to a circle domain, i.e., a domain whose boundary components are either points or circles. Koebe (1920) proved the conjecture for all finitely connected domains.	Due to Grötzsch (1931) and de Possel (1932), this uniformization says that every plane domain admits a conformal map onto a parallel slit domain. A parallel slit domain is a domain all of whose complementary components are parallel slits, i.e., straight line segments inclined at a fixed angle, or points.

Circle domains

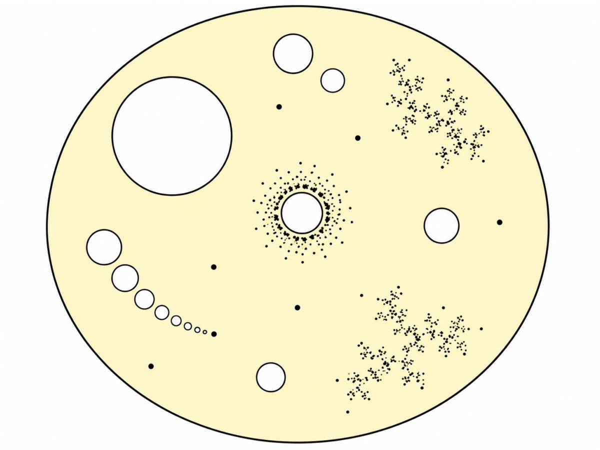

Figure 1. A circle domain. Its boundary may contain isolated circles, circles converging to points, isolated points, points converging to circles, Cantor sets, etc.

Note that there are at most countably many circular boundary components, whereas the point components may form a very large set; for instance, a positive-area Cantor set may occur in the boundary.

Development on Koebe's conjecture

Types of domains	**Existence**	**Uniqueness**
Simply connected	✓ Riemann (1851)	✓ Riemann (1851)
Finitely connected	✓ Koebe (1920)	✓ Koebe (1920)
Countably connected	✓ He–Schramm (1993)	✓ He–Schramm (1993)
Uniform	✓ Herron–Koskela (1990)	✓ Ntalampekos–Younsi (2020)
Cofat	✓ Schramm (1995)	× positive-area Cantor set
Cospread	✓ Esmayli–Rajala (2026)	× positive-area Cantor set
Gromov hyperbolic	✓ Karafyllia–Ntalampekos (2025)	✓ Ntalampekos–Younsi (2020)

- For a domain $G \subset \widehat{\mathbb{C}}$, let $\mathcal{C}(G) = \mathcal{C}_N(G) \cup \mathcal{C}_P(G)$ be the collection of connected components of $\widehat{\mathbb{C}} \setminus G$, where $\mathcal{C}_N(G)$ denotes the non-point complementary components and $\mathcal{C}_P(G)$ denotes the point-components.
- Let $\widehat{G} = \widehat{\mathbb{C}}/\sim$, where $x \sim y$ if either $x = y \in G$, or $x, y \in a$ for some $a \in \mathcal{C}(G)$. We equip $\widehat{G}$ with the quotient topology and denote the quotient map by $\pi_G : \widehat{\mathbb{C}} \longrightarrow \widehat{G}$.
- By Moore's theorem, the quotient space $\widehat{G}$ is homeomorphic to $\widehat{\mathbb{C}}$. Hence, if $f : G \to G'$ is a homeomorphism, then f naturally extends to a homeomorphism $\widehat{f} : \widehat{G} \to \widehat{G'}$.

Strong Koebe-type conclusion

Find a conformal homeomorphism $f : \Omega \to D$ onto a circle domain D such that

$$\widehat{f}(\mathcal{C}_N(\Omega)) = \mathcal{C}_N(D), \qquad \widehat{f}(\mathcal{C}_P(\Omega)) = \mathcal{C}_P(D). \tag{1}$$

- He–Schramm gives circle-domain uniformization for countably connected domains, but not necessarily preservation of point and non-point components. Indeed, this stronger conclusion does not hold for all countably connected domains.
- Esmayli–Rajala construct a countably connected domain Ω with $\{0\} \in \mathcal{C}_P(\Omega)$ such that for every conformal homeomorphism $f : \Omega \to D$ onto a circle domain, $\mathrm{diam}(\widehat{f}(\{0\})) > 0$. Thus a point-component of Ω becomes a non-point component of D.
- Therefore extra assumptions are needed for the strong Koebe conclusion

Schramm's cofat uniformization theorem

Definition

- A compact set $A \subset \widehat{\mathbb{C}}$ is called *τ-fat*, $\tau > 0$, if for every $z \in A \cap \mathbb{C}$ and every Euclidean ball $B(z, r)$ which does not contain A, one has

$$\mathrm{Area}(A \cap B(z, r)) \geq \tau r^2.$$

- A domain $\Omega \subset \widehat{\mathbb{C}}$ is called *cofat* if there exists $\tau > 0$ such that every connected component of $\widehat{\mathbb{C}} \setminus \Omega$ is τ-fat.

- For non-degenerate compact planar sets, fatness is essentially Ahlfors 2-regularity.
- Cofatness is an area-thickness condition on the non-trivial complementary components. Point components are fat vacuously.
- Non-trivial complementary components must have positive area at all relevant scales.

Theorem (Schramm, 1995)

Every cofat domain $\Omega \subset \widehat{\mathbb{C}}$ is conformally equivalent to a circle domain with the Strong Koebe conclusion (1).

- In the same work Schramm introduced the notion of transboundary modulus and used it, in combination with cofat domains, to provide an alternative proof of the He–Schramm uniformization theorem for countably connected domains.
- Transboundary modulus allows curves to interact with complementary components. It became a central tool in later uniformization and rigidity problems.
- Esmayli–Rajala's work also relies on Schramm's transboundary modulus framework.

Definition of quasitripod

Let

$$T_0 = \bigcup_{k=0}^{2} [0, e^{2\pi i k/3}]$$

be the standard tripod. A set $T \subset \mathbb{C}$ is called a H*-quasitripod* if it is a weakly H-quasisymmetric image of T_0.

Main Theorem

Theorem A: Esmayli-Rajala (2024)

Let $\Omega \subset \widehat{\mathbb{C}}$ be a domain containing ∞. Suppose that there are constants $H, N \geq 1$ such that:

(i) Branching condition: every $p \in \mathcal{C}_N(\Omega)$ contains an H-quasitripod T with $\operatorname{diam}(T) \geq \operatorname{diam}(p)/H$;

(ii) Packing condition: for every $z_0 \in \mathbb{C}$ and $r > 0$,

$$\operatorname{card}\{p \in \mathcal{C}_N(\Omega) : \operatorname{diam}(p) \geq r,\ p \cap \mathbb{D}(z_0, r) \neq \emptyset\} \leq N.$$

Then Ω is conformally equivalent to a circle domain D with the strong Koebe conclusion (1).

Theorem A assumes two things:

(i) Every non-point complementary component contains a large quasitripod.

(ii) At every location and scale, there are at most N large non-point components nearby. This packing condition implies that $\mathcal{C}_N(\Omega)$ is countable.

- Esmayli–Rajala then introduce a natural class of domains for which the hypotheses of Theorem A are verified: cospread domains.

Definition: Cospread domains

- A compact set $A \subset \widehat{\mathbb{C}}$ is called *H-spread* if for every $z \in A \cap \mathbb{C}$ and every $0 < r < \operatorname{diam}(A \cap \mathbb{C})$ there exists an H-quasitripod $T \subset A \cap B(z, r)$ with $\operatorname{diam}(T) \geq r/H$.
- A domain $\Omega \subset \widehat{\mathbb{C}}$ is *cospread* if every $p \in \mathcal{C}_N(\Omega)$ is H-spread for some $H \geq 1$.

- Cospreadness is a local version of Condition (i): large quasitripods occur at every point and every scale.
- Esmayli–Rajala proved that this local branching condition also forces the packing condition (ii). That is

Proposition: Esmayli–Rajala (2024)

If Ω is H-cospread, then Conditions (i) and (ii) of Theorem A hold, with constants depending only on H.

Hence, cospread domains admit conformal maps onto circle domains with strong Koebe conclusion (1).

Cofat versus cospread domains

	Cofat domains	**Cospread domains**
Geometric nature	Area thickness of complementary components	Branching of complementary components at all locations and scales
Local condition	$\mathrm{Area}(A \cap B(z,r)) \gtrsim r^2$	$A \cap B(z,r)$ contains a large quasitripod
Non-point components	Must have positive area	May be very thin, even of zero area
Examples	Quasidisks; uniform domains	Complements of spread Julia sets, e.g. $\widehat{\mathbb{C}} \setminus J(z^2+i)$; complements of self-similar or uniformly branching trees
Non-examples	Outward cusp; slit-like complementary component	Slit or arc component, e.g. $\widehat{\mathbb{C}} \setminus [0,1]$
Invariance	Möbius and quasiconformal invariant	Quasi-Möbius invariant; hence Möbius and quasiconformal invariant

Proof of Theorem A

- The proof of Theorem A requires transboundary modulus estimates which are significantly more involved than the corresponding estimates for cofat domains.
- In Schramm's cofat theorem, the complementary components are area-thick. This gives local ℓ^2-control of their diameters:

$$\sum_a \operatorname{diam}(\overline{a} \cap B(w,R))^2 \lesssim_\tau R^2.$$

  This estimate is extremely useful in transboundary modulus arguments.
- Unlike cofatness, the conditions in Esmayli–Rajala's theorem do not imply ℓ^2-control of the diameters of the elements of $\mathcal{C}_N(\Omega)$. The components may be very thin and may have zero area; for example, the Julia set of $z^2 + i$ is an η-spread set of zero area.
- Therefore the proof of Theorem A needs more delicate transboundary modulus estimates.

Transboundary modulus: extended metrics

Let $G \subset \widehat{\mathbb{C}}$ be a domain. Recall that $\widehat{G} = \widehat{\mathbb{C}}/\sim$, where each complementary component of G is collapsed to one point.

Extended metrics

- The area measure $\mu = \mu_G$ on $\widehat{G}$ is defined by

$$\mu|_{\pi_G(G)} = \text{Lebesgue area measure on } G \text{ and}$$

$$\mu|_{\mathcal{C}(G)} = \text{counting measure.}$$

- An *extended metric* on $\widehat{G}$ is a Borel function $\rho : \widehat{G} \to [0, \infty]$. Its area is

$$A(\rho) = \int_G (\rho \circ \pi_G)^2 \, dA + \sum_{p \in \mathcal{C}(G)} \rho(p)^2.$$

Transbounday length and modulus

Transboundary length

Let γ be a path in $\widehat{G}$. Its transboundary ρ-length is

$$\ell_\rho(\gamma) = \int_{\gamma \cap \pi_G(G)} \rho \, ds + \sum_{\substack{p \in \mathcal{C}(G) \\ p \in |\gamma|}} \rho(p).$$

The first term is computed on the parts of γ lying in $\pi_G(G)$, lifted back to G using $(\pi_G|_G)^{-1} : \pi_G(G) \to G$.

Transboundary modulus

Let Γ be a family of paths in $\widehat{G}$. An extended metric ρ is *admissible* for Γ if $\ell_\rho(\gamma) \geq 1$ for every $\gamma \in \Gamma$.

Then

$$\operatorname{mod}_G(\Gamma) := \inf_\rho \left\{ \int_G (\rho \circ \pi_G)^2 \, dA + \sum_{p \in \mathcal{C}(G)} \rho(p)^2 \right\},$$

where the infimum is taken over all admissible extended metrics.

Basic properties

Lemma: conformal invariance

Suppose that $f : G \to G'$ is conformal. Then for every path family Γ in $\widehat{G}$, $\mathrm{mod}_G(\Gamma) = \mathrm{mod}_{G'}(\widehat{f}(\Gamma))$, where $\widehat{f}(\Gamma) := \{\widehat{f} \circ \gamma : \gamma \in \Gamma\}$.

Basic properties of transboundary modulus

Let $\Gamma_1, \Gamma_2, \ldots$ be path families in $\widehat{G}$.

1. If $\Gamma_1 \subset \Gamma_2$, then $\mathrm{mod}_G(\Gamma_1) \leq \mathrm{mod}_G(\Gamma_2)$.
2. If $\Gamma = \bigcup_j \Gamma_j$, then $\mathrm{mod}_G(\Gamma) \leq \sum_j \mathrm{mod}_G(\Gamma_j)$.
3. If Γ_1 minorizes Γ_2, meaning every $\gamma_2 \in \Gamma_2$ contains a subpath $\gamma_1 \in \Gamma_1$, then $\mathrm{mod}_G(\Gamma_1) \geq \mathrm{mod}_G(\Gamma_2)$.
4. If $p \in \pi_G(G)$, or if p is an isolated point-component, then the modulus of all paths γ in $\widehat{G}$ satisfying $p \in |\gamma|$ is zero.

Proof idea: the key estimate

Theorem B: Esmayli–Rajala

Let $\Omega \subset \widehat{\mathbb{C}}$ be a finitely connected domain satisfying Conditions (i) and (ii) of Theorem A with constants H, N. Then there is $M = M(H, N) > 0$ such that for every $a \in \mathbb{C}$ and $R > 0$, $\operatorname{mod}_\Omega(\Gamma) \leq M$, where Γ is the family of paths in $\pi_\Omega(A(a, R))$ joining $\pi_\Omega(S(a, 4R))$ and $\pi_\Omega(S(a, R/2))$, with $A(a, R) = \mathbb{D}(a, 4R) \setminus \overline{\mathbb{D}(a, R/2)}$.

- This is the technical core of the paper.
- It replaces Schramm's cofat annulus estimate.
- We are now going to discuss how Theorem B can be applied to prove Theorem A.

Step 1: approximate by finitely connected domains

- By the packing condition, the non-point complementary components are countable: $\mathcal{C}_N(\Omega) = \{p_0, p_1, p_2, \ldots\}$.
- After filling in the point-components, it suffices to consider the domain whose complementary components are precisely the elements of $\mathcal{C}_N(\Omega)$.
- For $k \geq 0$, define $\widetilde{\Omega}_k = \widehat{\mathbb{C}} \setminus (p_0 \cup \cdots \cup p_k)$. Then $\widetilde{\Omega}_k$ is finitely connected and $\widetilde{\Omega}_k \supset \Omega$.
- By Koebe's theorem, there are conformal maps $g_k : \widetilde{\Omega}_k \to \widetilde{D}_k$, where each $\widetilde{D}_k$ is a circle domain. After normalization and passing to a subsequence, $g_k \to f$ locally uniformly on Ω.
- Hence, for each fixed ℓ and all $k \geq \ell$, the image of the complementary component p_ℓ is a closed round disk in $\widehat{\mathbb{C}}$: $q_{k,\ell} := \widehat{g}_k(p_\ell)$.
- After passing to a diagonal subsequence, for every fixed ℓ, $q_{k,\ell} \to q_\ell$ in the Hausdorff sense. Thus q_ℓ is either a closed round disk or a point.

Step 2: what can go wrong in the limit?

- We do not know whether $D = f(\Omega)$ is a circle domain. It could be possible that
  - if $p \in \mathcal{C}_P(\Omega)$, then $\widehat{f}(p)$ is neither a point nor a closed round disk;
  - if $p_\ell \in \mathcal{C}_N(\Omega)$, then $\widehat{f}(p_\ell) \neq q_\ell$.
- Since we also need the strong Koebe conclusion (1), we must also rule out the possibility that a non-point component p_ℓ collapses to a point.

Thus, to prove that $D = f(\Omega)$ is a circle domain and (1) holds, we show that

$$p \in \mathcal{C}_P(\Omega) \implies \operatorname{diam}\big(\widehat{f}(p)\big) = 0, \tag{2}$$

$$p_\ell \in \mathcal{C}_N(\Omega) \implies \widehat{f}(p_\ell) = q_\ell, \tag{3}$$

$$p_\ell \in \mathcal{C}_N(\Omega) \implies \operatorname{diam}(q_\ell) > 0. \tag{4}$$

Step 3: the relevant path families

Fix a complementary component $\bar{p} \in \mathcal{C}(\Omega)$ and a Jordan curve $J \subset \Omega$ separating $\bar{p}$ from the rest. Esmayli–Rajala define two curve families:

- Γ_j: paths joining the fixed curve J to a small segment approaching $\bar{p}$;
- Λ_j: paths separating J from $\bar{p}$.

These families are chosen so that Theorem B gives the required upper bounds, while the circle-domain geometry of D_j gives lower bounds for the image families $\widehat{f}_j(\Gamma_j)$ and $\widehat{f}_j(\Lambda_j)$.

Step 4: circle-domain estimates

Esmayli–Rajala proved the following circle-domain estimates.

Circle-domain estimates

Let $f_j : \Omega_j \to D_j$ be the conformal maps defined above, where each D_j is a circle domain. The following estimates hold:

1. There is a homeomorphism $\varphi_{\bar{p}} : [0, \infty) \to [0, \infty)$ such that $\limsup_{j\to\infty} \operatorname{mod}_{D_j} \widehat{f}_j(\Gamma_j) \geq \limsup_{j\to\infty} \varphi_{\bar{p}}(\operatorname{dist}(f_j(b), \widehat{f}_j(\bar{p})))$.
2. If $\operatorname{diam}(\widehat{f}(\bar{p})) = 0$, then $\lim_{j\to\infty} \operatorname{mod}_{D_j} \widehat{f}_j(\Lambda_j) = \infty$.

Summary of the modulus contradiction

Proof of (2), (3), and (4)

- Iterating Theorem B over a sequence of disjoint annuli gives a vanishing upper bound $\mathrm{mod}_{\Omega_j}(\Gamma_j) \leq \theta_a(r)$, where $\theta_a(r) \to 0$ as $r \to 0$.
- Theorem B also gives a finite upper bound $\mathrm{mod}_{\Omega_j}(\Lambda_j) \leq M_\ell < \infty$.
- The circle-domain estimates give lower bounds for $\mathrm{mod}_{D_j}\, \widehat{f}_j(\Gamma_j)$ and $\mathrm{mod}_{D_j}\, \widehat{f}_j(\Lambda_j)$.
- Since transboundary modulus is conformally invariant, $\mathrm{mod}_{\Omega_j}(\Gamma_j) = \mathrm{mod}_{D_j}\, \widehat{f}_j(\Gamma_j)$ and $\mathrm{mod}_{\Omega_j}(\Lambda_j) = \mathrm{mod}_{D_j}\, \widehat{f}_j(\Lambda_j)$. Combining the upper and lower bounds gives the desired contradictions.

Proof of (2)

Suppose, towards a contradiction, that $p = \{a\} \in \mathcal{C}_P(\Omega)$ but $\widehat{f}(p)$ is a non-point component.

- Then there exist $c > 0$ and points $b_m \in \Omega$ with $b_m \to a$ such that $\limsup_{j\to\infty} \operatorname{dist}(f_j(b_m), \widehat{f_j}(p)) \geq c$ for every m.
- The circle-domain estimate gives $\limsup_{j\to\infty} \operatorname{mod}_{D_j} \widehat{f_j}(\Gamma_j) \geq c_0 > 0$.
- On the other hand, iterating Theorem B over disjoint annuli gives $\operatorname{mod}_{\Omega_j}(\Gamma_j) \leq \theta_a(|b_m - a|)$, where $\theta_a(r) \to 0$ as $r \to 0$.
- By conformal invariance, $\operatorname{mod}_{\Omega_j}(\Gamma_j) = \operatorname{mod}_{D_j} \widehat{f_j}(\Gamma_j)$, which gives a contradiction as $m \to \infty$.

Therefore point-components remain point-components.

Proof of (3) and (4)

Let $p_\ell \in \mathcal{C}_N(\Omega)$.

- By Carathéodory kernel convergence, one first obtains $q_\ell \subset \widehat{f}(p_\ell)$.
- If the inclusion were strict, then one could find points $b_m \in \Omega$ with $\mathrm{dist}(b_m, p_\ell) \to 0$ but whose images stay a definite distance from $\widehat{f}_j(p_\ell)$.
- The same Γ_j-argument as before gives a contradiction. Hence $q_\ell = \widehat{f}(p_\ell)$.
- Now suppose, towards a contradiction, that q_ℓ is a point. Then $\mathrm{diam}(\widehat{f}(p_\ell)) = 0$.
- By the circle-domain estimate, $\mathrm{mod}_{D_j}\, \widehat{f}_j(\Lambda_j) \to \infty$.
- But Theorem B gives the finite upper bound $\mathrm{mod}_{\Omega_j}(\Lambda_j) \leq M_\ell < \infty$.
- Again conformal invariance gives a contradiction. Hence $\mathrm{diam}(q_\ell) > 0$.

Conclusion of the proof

- We have shown: $\mathrm{diam}(\widehat{f}(p)) = 0$ for all $p \in \mathcal{C}_P(\Omega)$, and $q_\ell = \widehat{f}(p_\ell)$, $\mathrm{diam}(q_\ell) > 0$ for all $p_\ell \in \mathcal{C}_N(\Omega)$.
- Since $D = f(\Omega)$, every complementary component of D arises as the image, or limiting image, of a complementary component of Ω under the induced map on the decomposition compactifications.
- Therefore $\widehat{f}(\mathcal{C}_P(\Omega)) = \mathcal{C}_P(D)$ and $\widehat{f}(\mathcal{C}_N(\Omega)) = \mathcal{C}_N(D)$.
- Thus $D = f(\Omega)$ is a circle domain and f satisfies the strong Koebe-type conclusion.

Thank you