D. Šarić: Circle Homeomorphisms and Shears

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Geometric Function Theory Student Conference

6–10 July 2026

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Introduction

Main Goal

We give parameterizations of homeomorphisms, quasisymmetric maps, and symmetric maps of the unit circle in terms of shear coordinates for the Farey tessellation.

Introduction

Theorem A (Quasisymmetric & Symmetric Maps)

A shear map $s : F \to \mathbb{R}$ induces a quasisymmetric map iff $\exists M \geq 1$ such that for all fans $e_n \in F$ $(n \in \mathbb{Z})$, and all $m, k \in \mathbb{Z}$:

$$\frac{1}{M} \leq \frac{e^{s_m/2} + e^{s_m/2+s_{m+1}} + \cdots + e^{s_m/2+s_{m+1}+\cdots+s_{m+k}}}{e^{-s_m/2} + e^{-s_m/2-s_{m-1}} + \cdots + e^{-s_m/2-s_{m-1}-\cdots-s_{m-k}}} \leq M$$

It induces a symmetric map if this ratio tends to 1 as Farey generations go to infinity.

Theorem B (Teichmüller Topology)

Let $h_n, h \in \mathrm{Möb}(S^1)\backslash\mathrm{QS}(S^1)$.
Then $h_n \to h$ in the Teichmüller topology iff $\sup_p M_{s,s_n}(p) \to 1$ as $n \to \infty$.

Theorem C (Homeomorphisms)

A shear map $s : F \to \mathbb{R}$ induces a homeomorphism iff for each chain of edges $e_n \in F$ $(n \in \mathbb{N})$:

$$\sum^{\infty} e^{s_1^n+s_2^n+\cdots+s_n^n} = \infty \quad (\text{where } s_i^n = \pm s(e_i))$$

Introduction

Theorem D (Homeomorphisms via Lambda Lengths)

A lambda length function $\lambda : F \to \mathbb{R}^+$ induces a homeomorphism iff for each chain of edges $e_n \in F$ ($n \in \mathbb{N}$):

$$\sum_{n=1}^{\infty} \left[\lambda_n^{-1/2} \lambda_{n-1}^{1/2} \dots \lambda_1^{(-1)^n/2} \right] \alpha_n = \infty$$

where $\lambda_i = \lambda(e_i)$ and α_n is the horocyclic length of the wedge bounded by e_n and e_{n+1}.

Theorem E (Quasisymmetric & Symmetric Maps)

A lambda length function $\lambda : F \to \mathbb{R}^+$ induces a quasisymmetric map iff $\exists K \geq 1$ such that for all fans $e_n \in F$, $m \in \mathbb{Z}$, and $k \in \mathbb{N}$:

$$\frac{1}{K} \leq \frac{\alpha(e_m, e_{m+1}) + \alpha(e_{m+1}, e_{m+2}) + \dots + \alpha(e_{m+k}, e_{m+k+1})}{\alpha(e_m, e_{m-1}) + \alpha(e_{m-1}, e_{m-2}) + \dots + \alpha(e_{m-k}, e_{m-k-1})} \leq K$$

It induces a symmetric map if this ratio tends to 1 as Farey generations go to infinity.

Quasisymmetric Maps and Barycentric Extension

Definition (M-Quasisymmetric Map)

A homeomorphism $h : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ fixing ∞ is M-quasisymmetric for $M \geq 1$ iff $\forall x \in \mathbb{R}$ and $t > 0$:

$$\frac{1}{M} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq M$$

Universal Teichmüller Space Topology

$T(\mathbb{H})$ consists of quasisymmetric maps fixing $0, 1, \infty$.
Convergence $h_n \to h \in T(\mathbb{H})$ in the Teichmüller topology means:

$$h_n \circ h^{-1} \to \mathrm{id} \quad (\text{as } M_n \to 1 \text{ for } n \to \infty)$$

Quasisymmetric Maps and Barycentric Extension

Barycentric Extension Properties

The Douady-Earle barycentric extension $\mathrm{ex}(h) : \mathbb{H} \to \mathbb{H}$ is conformally natural:

$$\mathrm{ex}(A \circ h \circ B) = A \circ \mathrm{ex}(h) \circ B \quad \forall A, B \in \mathrm{PSL}_2(\mathbb{R})$$

If $h_n \to h$ pointwise, then the Beltrami coefficients converge uniformly on compact subsets:

$$\mu(\mathrm{ex}(h_n)) \to \mu(\mathrm{ex}(h))$$

Lemma 2.2 (Markovic, Douady-Earle, Abikoff-Earle-Mitra)

Let h_n fix $0, 1, \infty$. If $\exists c_0 \geq 1$ such that $-c_0 \leq h_n(-1) \leq -1/c_0$, then $\forall n$, there is a neighborhood U of $i \in \mathbb{H}$ and $0 < c < 1$ such that:

$$\|\mu_n|_U\|_\infty \leq c < 1$$

Farey Tessellation and Shear Map

Farey Tessellation & Generations

The Farey tessellation F is the Γ-orbit of the boundary sides of the ideal triangle $\Delta_0(0, 1, \infty)$ under hyperbolic reflections. The vertex set is $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$.

- **Generation 0:** Boundary edges of Δ_0.
- **Generation n:** Edges obtained by a minimum of n reflections.

Geometric Definition of Shear

Let (Δ_1, Δ_2) share an edge. Let $A \in \mathrm{PSL}_2(\mathbb{R})$ map $\Delta_1 \to (-1, 0, \infty)$ and the common edge $\to (0, \infty)$. Then $A(\Delta_2)$ has vertices $(0, e^r, \infty)$, where:

$$\text{Shear of } (\Delta_1, \Delta_2) = r \in \mathbb{R}$$

Any two adjacent complementary triangles in F have shear 0.

Farey Tessellation and Shear Map

Definition 3.1 (The Shear Map)

A homeomorphism $h : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ extends to the space of geodesics G. Each edge $e \in F$ bounds exactly two triangles Δ_1, Δ_2. The shear map $s_h : F \to \mathbb{R}$ assigns to e the shear of the image pair:

$$s_h(e) = \text{shear}(h(\Delta_1), h(\Delta_2))$$

Uniqueness & Characteristic Map

If a homeomorphism $h : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ fixes $0, 1, \infty$, it is uniquely determined by its shear map s_h. Given any shear map $s : F \to \mathbb{R}$, there exists a unique injective characteristic map:

$$h_s : \hat{\mathbb{Q}} \to \hat{\mathbb{R}} \quad (\text{fixing } 0, 1, \infty)$$

Homeomorphisms and Shears

Piecewise Möbius Cocycle Map H_s

An arbitrary map $s : F \to \mathbb{R}$ induces a cocycle map $H_s : \mathbb{H} \to \mathbb{H}$. Set $H_s|_{\Delta_0} = \mathrm{id}$. For any other complementary triangle $\Delta \in \Gamma(\Delta_0)$, let $\{e_1, e_2, \ldots, e_n\}$ be the ordered intersecting edges of a geodesic arc from the center of Δ_0 to Δ:

$$H_s|_\Delta = T_{e_1}^{s(e_1)} \circ T_{e_2}^{s(e_2)} \circ \cdots \circ T_{e_n}^{s(e_n)}$$

where $T_{e_i}^{s(e_i)}$ is the hyperbolic translation along oriented axis e_i. This extends to a monotone characteristic map $h_s : \hat{\mathbb{Q}} \to \hat{\mathbb{R}}$.

Proposition 4.1 (Surjectivity Criterion)

The characteristic map $h_s : \hat{\mathbb{Q}} \to \hat{\mathbb{R}}$ extends by continuity to a homeomorphism of $\hat{\mathbb{R}}$ if and only if the cocycle map $H_s : \mathbb{H} \to \mathbb{H}$ is surjective.

Homeomorphisms and Shears

Proposition 4.2 (Horocyclic Leaf Length)

Let W_n be the hyperbolic wedge bounded by $h_s(e_n)$ and $h_s(e_{n+1})$, foliated by horocyclic arcs. The map h_s continuously extends to $x \in \hat{\mathbb{R}} \setminus \hat{\mathbb{Q}}$ iff the horocyclic leaf $l(P_1)$ spanning $\bigcup_n W_n$ has infinite length:

$$\text{Total Length } L = \sum_{n=1}^{\infty} l_n = \sum_{n=1}^{\infty} e^{s_1^n + s_2^n + \cdots + s_n^n} = \infty$$

Inductive Leaf Calculation Rules (s_i^n Constants)

By choosing $l_1 = e^{s_1^1}$, the recursive wedge lengths l_{n+1} satisfy:

- **Same Fan:** $l_{n+1} = l_n e^{\pm s(e_{n+1})} \implies s_i^{n+1} = s_i^n$
- **Fan Change:** $l_{n+1} = l_n^{-1} e^{\pm s(e_{n+1})} \implies s_i^{n+1} = -s_i^n$ (for $i \leq n$)

This dynamic geometric progression explicitly proves Theorem C.

Quasisymmetric Maps and Shears

Normalized Fan Analysis at Infinity

By conformal naturality via $\mathrm{PSL}_2(\mathbb{R})$ mappings, a quasisymmetric boundary map h can be studied on a single normalized fan with its tip at ∞. The classical M-quasisymmetric condition evaluates symmetric triples in $\mathbb{Z}$:

$$\frac{1}{M} \leq \frac{h(n+k) - h(n)}{h(n) - h(n-k)} \leq M \quad \forall n \in \mathbb{Z},\ k \in \mathbb{N}$$

Geometric Necessity (Equation 5)

Normalizing h to fix $n, n+1$, and ∞ expresses the boundary increments completely via shear components $s_n = s(e_n)$ inside the targeted fan:

$$\frac{1}{M} \leq \frac{1 + e^{s_{n+1}} + \cdots + e^{s_{n+1}+s_{n+2}+\cdots+s_{n+k-1}}}{e^{-s_n} + e^{-s_n - s_{n-1}} + \cdots + e^{-s_n - s_{n-1} - \cdots - s_{n-k+1}}} \leq M$$

This bounds structural distortion across every adjacent ideal interval.

Quasisymmetric Maps and Shears

Symmetric Map Criteria & Convergence

A map is symmetric if its structural distortion approaches identity on fine scales. Over a fan-symmetric triple of high Farey generation, the shear values satisfy:

$$\lim_{\text{generation}\to\infty} \frac{1 + e^{s_1} + \cdots + e^{s_1+s_2+\cdots+s_n}}{e^{-s_0} + e^{-s_0-s_{-1}} + \cdots + e^{-s_0-s_{-1}-\cdots-s_{-n}}} = 1$$

Asymptotically Conformal Extension Proof

Sufficiency is proved by checking the Douady-Earle barycentric extension $F_s = \mathrm{ex}(h_s)$. If F_s fails to be quasiconformal, its Beltrami coefficient violates bounds:

$$|\mu_{F_s}(z_n)| \to 1 \quad \text{as } z_n \to \partial_\infty \mathbb{H}$$

Using a Cantor diagonalization over a dense set of dyadic rationals $D = \{k/2^{r-1}\}$, the rescaled maps $h_{n,r}$ pointwise converge to id, forcing $\mu \to 0$ and yielding a geometric contradiction.

The Topology on X

The Topology Metric $M(s_1, s_2)$

Given $s \in X$, $p \in \hat{\mathbb{Q}}$, and $e_n \in F$:

$$s(p; n, k) = \frac{\text{length}(h_s(e_{n+k}) \cap C, h_s(e_n) \cap C)}{\text{length}(h_s(e_n) \cap C, h_s(e_{n-k}) \cap C)}$$

$$M(s_1, s_2) = \sup_{p \in \hat{\mathbb{Q}},\, n,k \in \mathbb{Z}} \left\{ \max \left(\frac{s_1(p; n, k)}{s_2(p; n, k)}, \frac{s_2(p; n, k)}{s_1(p; n, k)} \right) \right\}$$

Theorem B (Teichmüller Convergence)

$h_n \to h \in \mathrm{M\ddot{o}b}(S^1) \backslash \mathrm{QS}(S^1)$ in Teichmüller topology $\iff$ $M(s_{h_n}, s_h) \to 1$ as $n \to \infty$.

- **Necessity:** $\sup \dfrac{\mathrm{cr}(h_n(a), h_n(b), h_n(c), h_n(d))}{\mathrm{cr}(a, b, c, d)} \to 1 \quad (\text{for } \mathrm{cr} = 2)$
- **Sufficiency:** $|\mu_F(z_n) - \mu_{F_n}(z_n)| \geq c > 0 \implies \lim_{m \to \infty} \tilde{h}'_{n_m, m} = \lim_{m \to \infty} \tilde{h}^*_{n_m, m}$ on $D = \{k/2^{r-1}\}$ at $i \in \mathbb{H}$.

Decorated Tessellations and Lambda Lengths

Geometric Parameters

- **Lambda length:** $\lambda(f) = e^{-2\delta(f)}$ ($\delta =$ signed horocycle distance)
- **Wedge length:** $\alpha(f, f') =$ horocyclic arc length between edges

Theorem E (Quasisymmetry)

Boundary map h_λ is quasisymmetric iff fan wedge sums are balanced:

$$\frac{1}{K} \leq \frac{\sum \alpha(\text{forward wedges})}{\sum \alpha(\text{backward wedges})} \leq K$$

Theorem D (Homeomorphism)

Boundary map h_λ is a homeomorphism iff every edge chain e_n satisfies:

$$\sum_{n=1}^{\infty} \left[\prod_{i=1}^{n} \lambda_i^{\frac{(-1)^{n-i+1}}{2}} \right] \alpha_n = \infty$$

Conclusion

Main Correspondence

Discrete Tessellation Geometry $\Longleftrightarrow$ Continuous Boundary Topology

Shear Maps

- **Homeomorphism:** Infinite horocyclic leaf accumulation.
- **Quasisymmetric:** Bounded structural distortion on all fans.
- **Symmetric:** Vanishing distortion at infinite generation.

Lambda Lengths

- **Equivalence:** Directly mirrors shear tracking.
- **Regularity:** Decoded completely through horocyclic wedge behavior and edge metric sequences.

Core Takeaway

The combinatorics of the Farey triangulation completely linearize the classification of the universal Teichmüller space.

Literature

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Thank You!